One-to-one correspondence between deterministic port-based teleportation and unitary estimation

Based on **arXiv:2408.11902**

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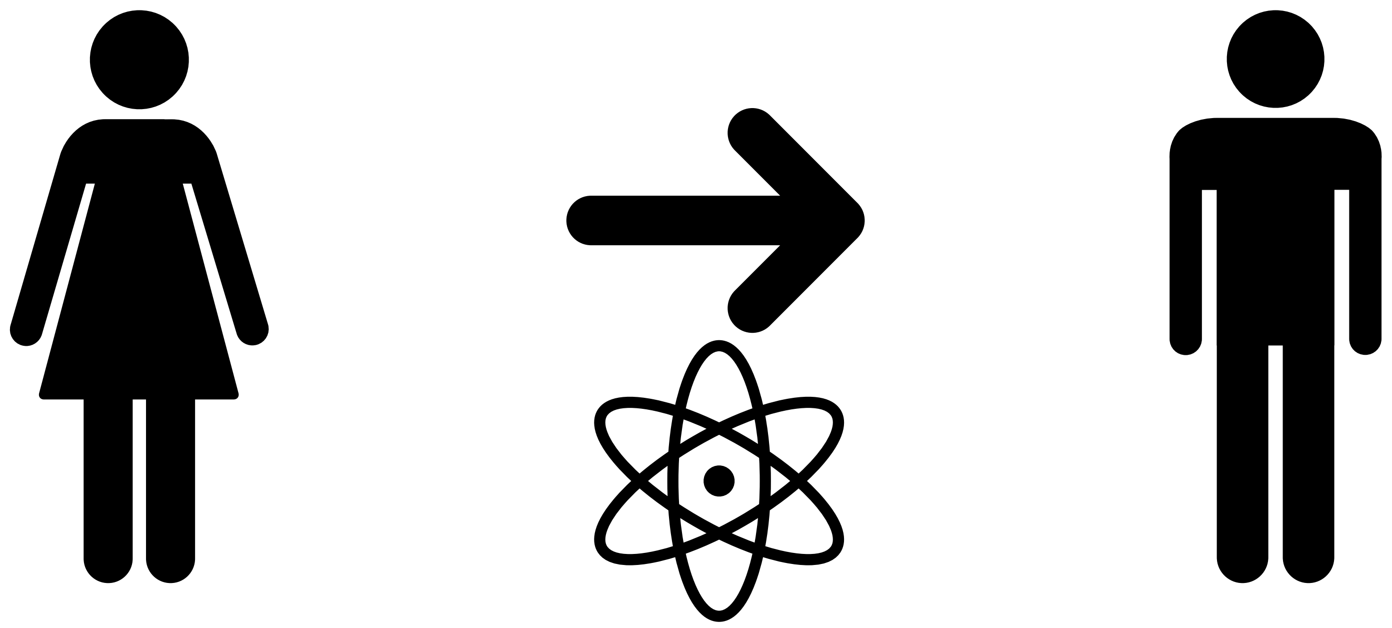
What this talk is about

Port-based teleportation

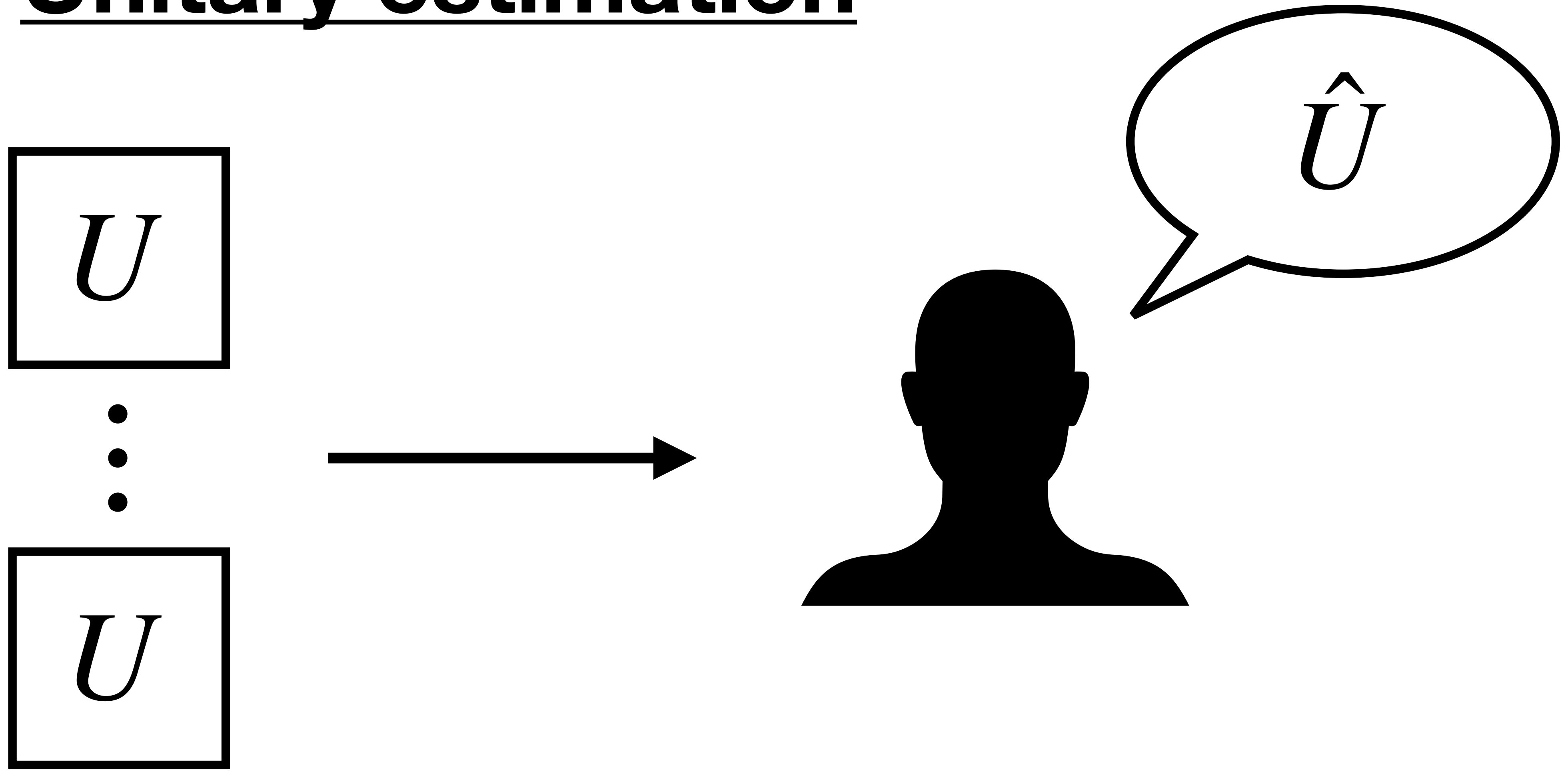


Teleportation fidelity F_{PBT}

$F_{\mathrm{PBT}} \mapsto F'_{\mathrm{est}} \geq F_{\mathrm{PBT}}$

$F_{\mathrm{est}} \mapsto F'_{\mathrm{PBT}} \geq F_{\mathrm{est}}$

Unitary estimation



Estimation fidelity F_{est}

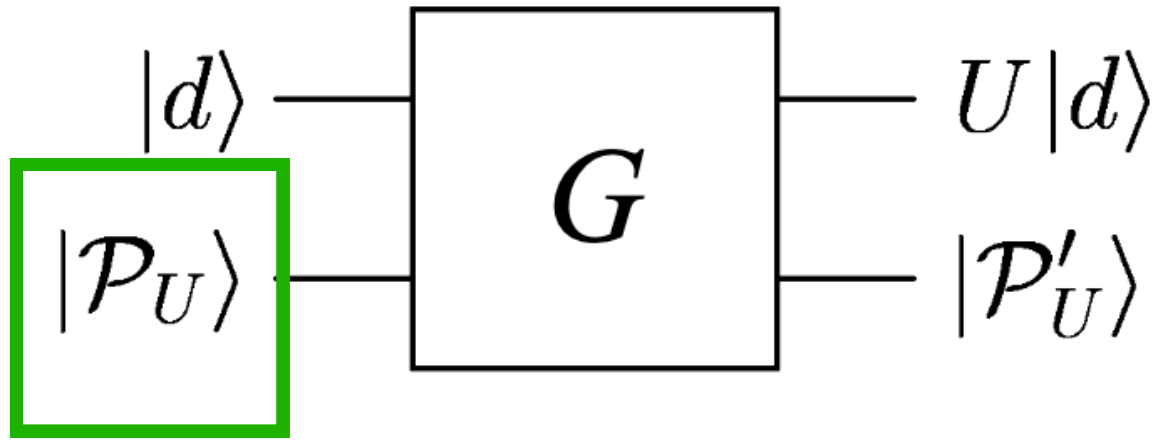
$\max F_{\mathrm{PBT}} = \max F_{\mathrm{est}}$

Constructive proof for the equivalence!

Port-based teleportation (PBT)

M. A. Nielsen and I. L. Chuang, PRL 79, 321 (1997)
M. Sedlák, A. Bisio, M. Ziman, PRL 122, 170502 (2019)

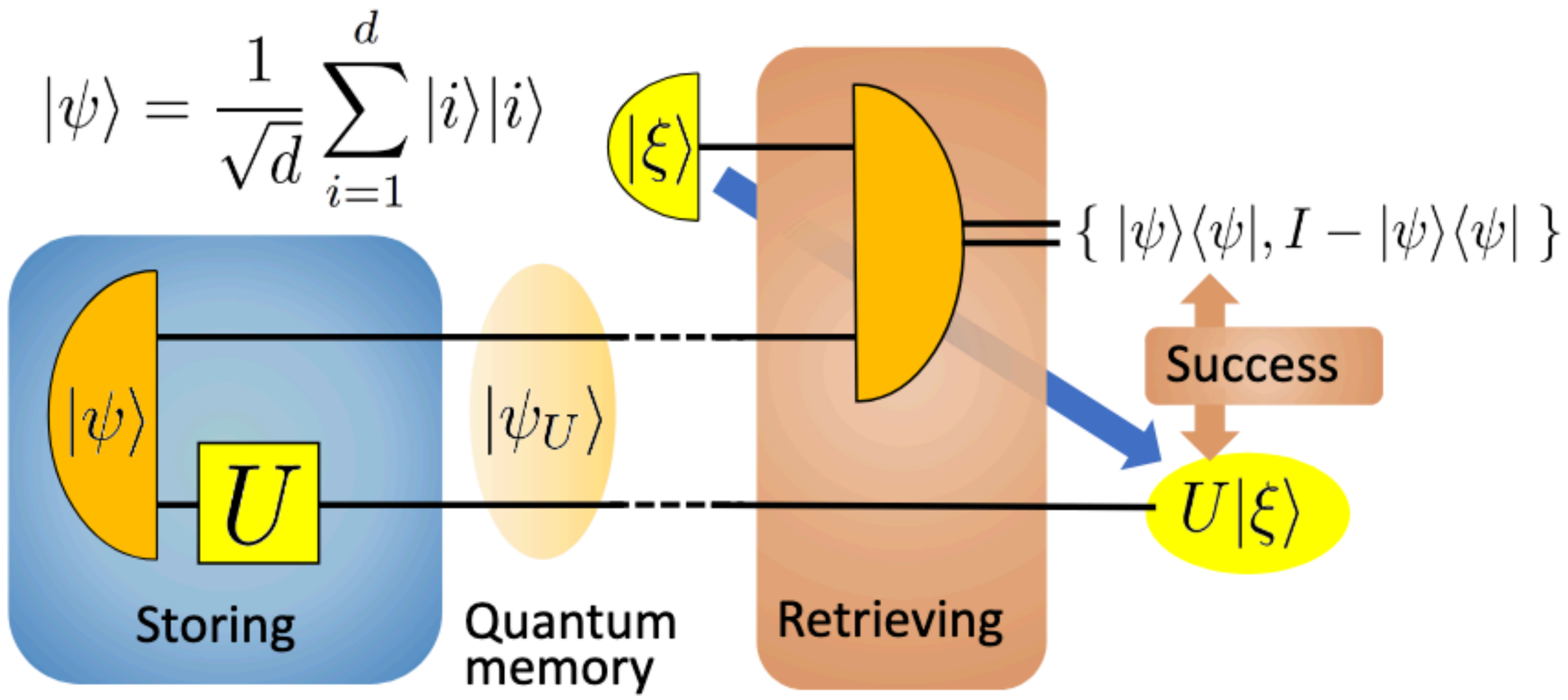
Universal programming of unitary operations



No-programming theorem: Universal programming is impossible

→ Probabilistic or non-exact implementation

Storage and retrieval (SAR) of quantum program via teleportation



If the Bell measurement outcome is (i, j), we get

$$UX^iZ^j|\xi\rangle$$

→ Retrieval succeeds when $(i, j) = (0,0)$

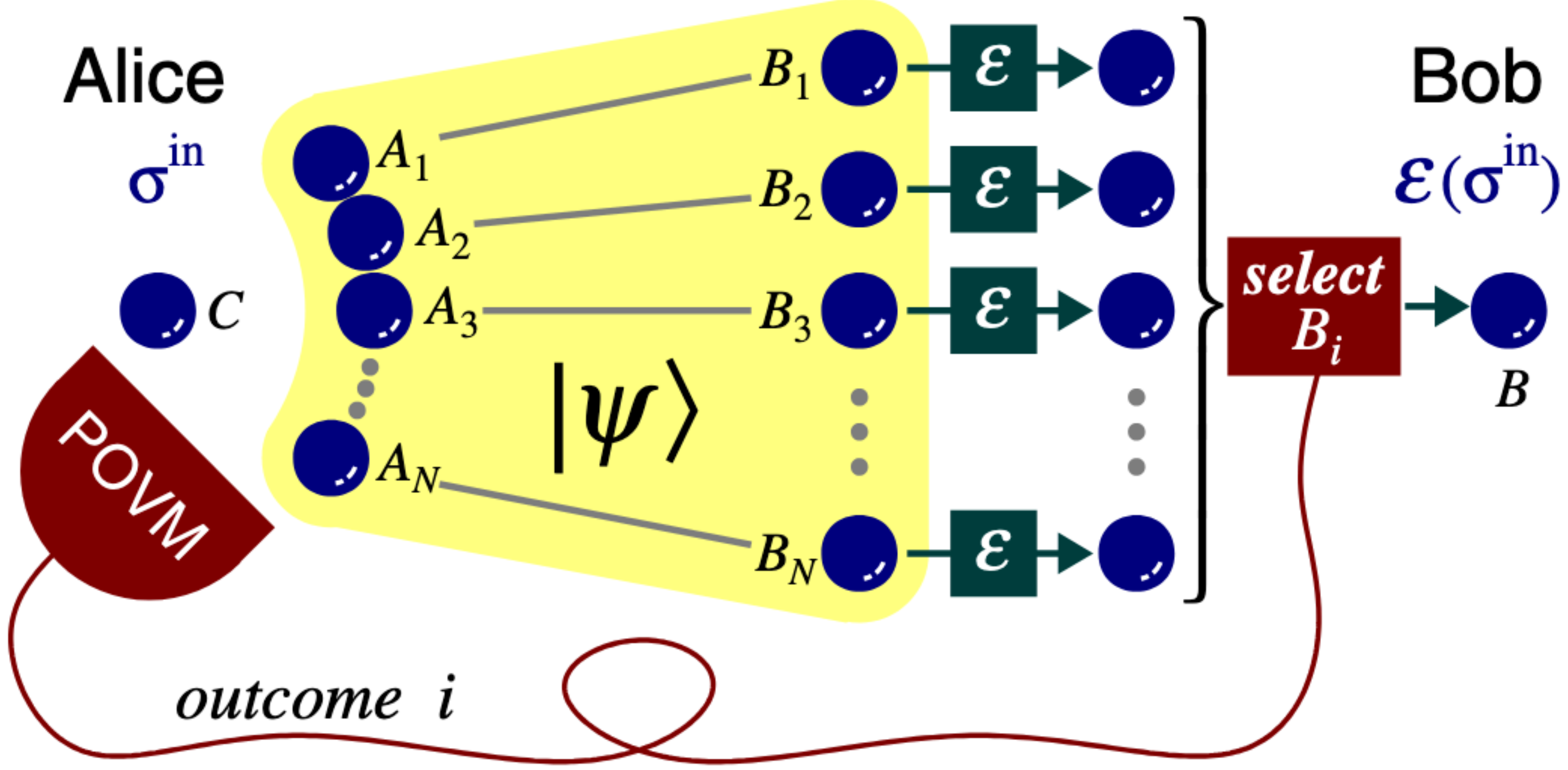
Success probability = $1/d^2$

Retrieval succeeds when Pauli correction is not needed

Port-based teleportation (PBT)

S. Ishizaka and T. Hiroshima, PRL 101, 240501 (2008)

SAR of quantum program via port-based teleportation (PBT)



1. Alice & Bob share $2N$-qudit entangled state $|\psi\rangle$
2. Alice measures input state σ^{in} with her share of $|\psi\rangle$
3. Alice sends the measurement outcome i to Bob
4. Bob selects i-th port **(no Pauli correction)**

Quantum state $(I^{\otimes N} \otimes \mathscr{E}^{\otimes N})(|\psi\rangle\langle\psi|)$ is the program state of $\mathscr{E}$

Port-based teleportation is quantum teleportation without Pauli correction

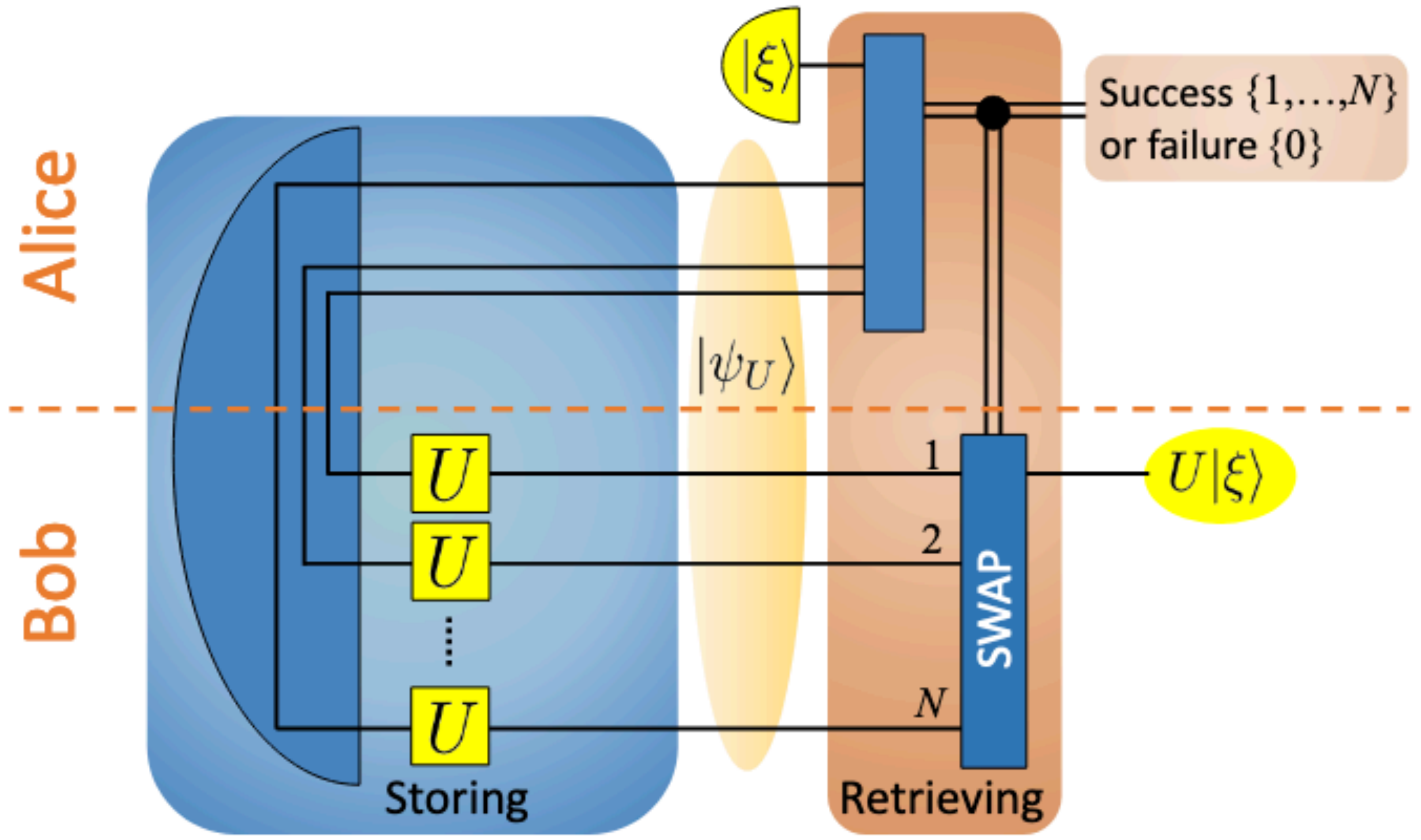
Port-based teleportation (PBT)

M. Sedlák, A. Bisio, M. Ziman, PRL 122, 170502 (2019)
A. Bisio et al. PRA 81, 032324 (2010)
Y. Yang, R. Renner, G. Chiribella, PRL 125, 210501 (2020)

Optimal protocols for SAR and PBT

Probabilistic exact (pPBT)

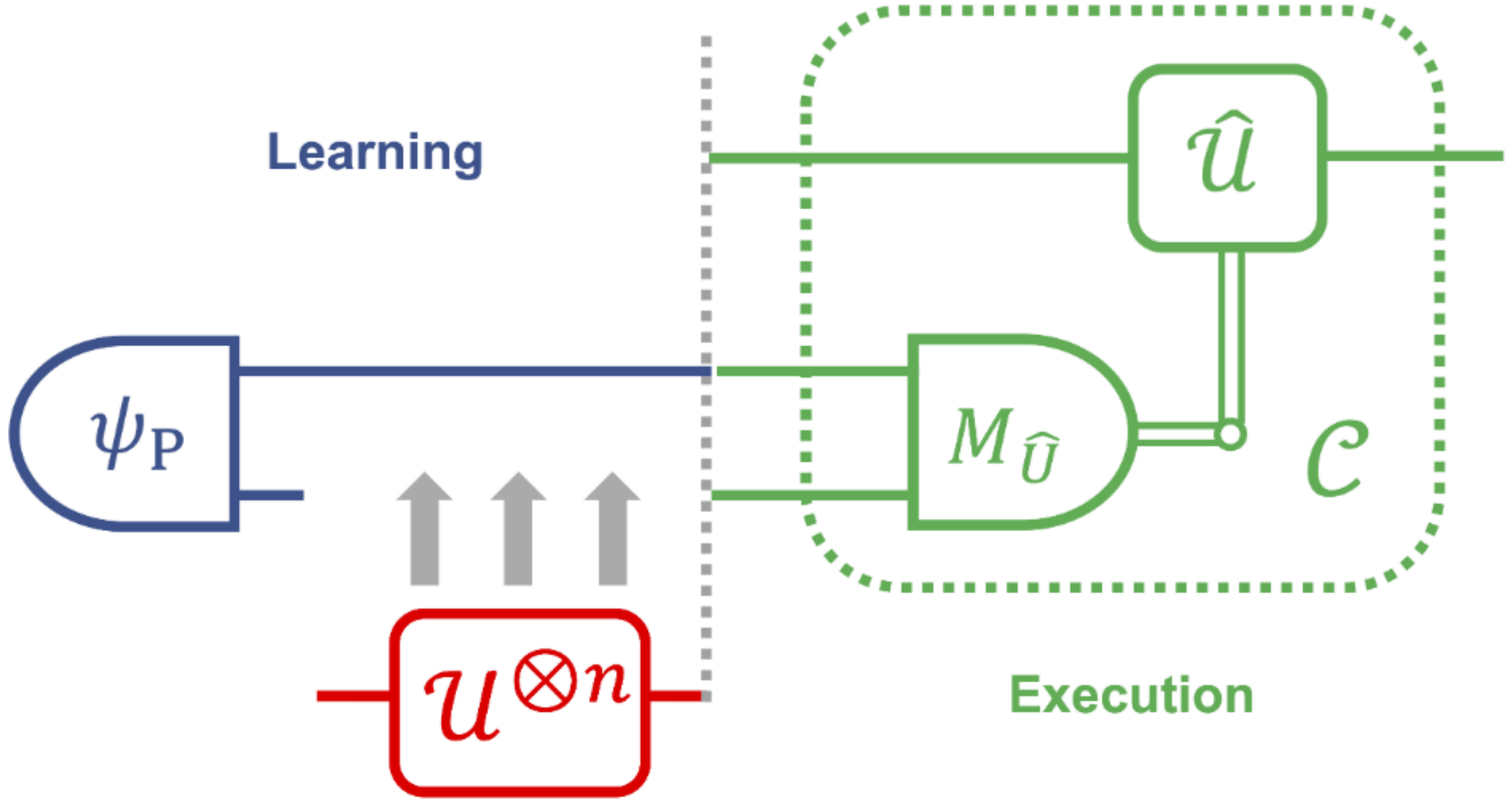
Optimal pSAR = pPBT



Optimal probability $p_{\mathrm{PBT}}(N,d) = N/(N-1+d^2)$

Deterministic non-exact (dPBT)

Optimal dSAR = unitary estimation ≠ dPBT



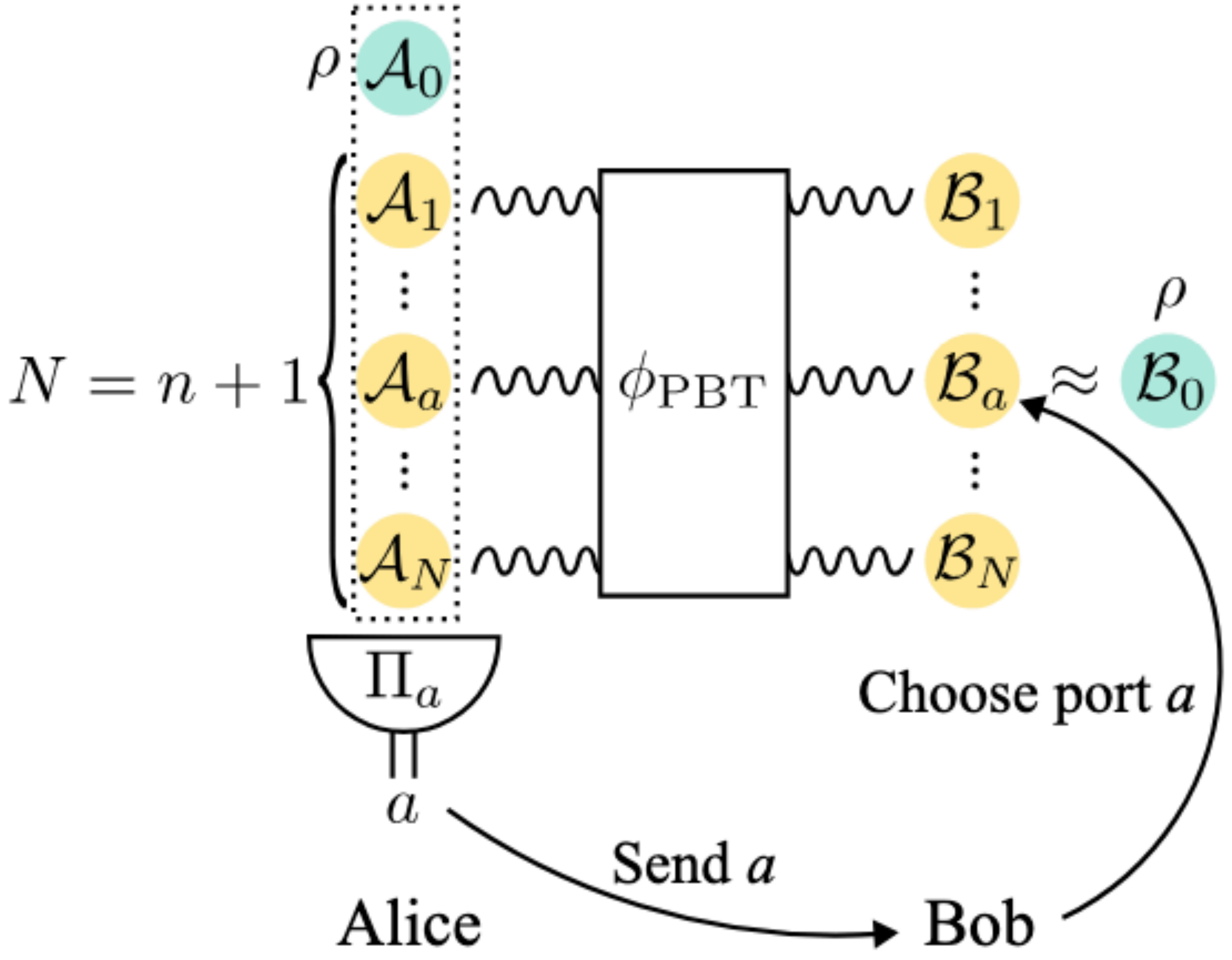
Optimal fidelity $F_{\mathrm{PBT}}(N,d) = ???$

Is there any connection between unitary estimation and dPBT?

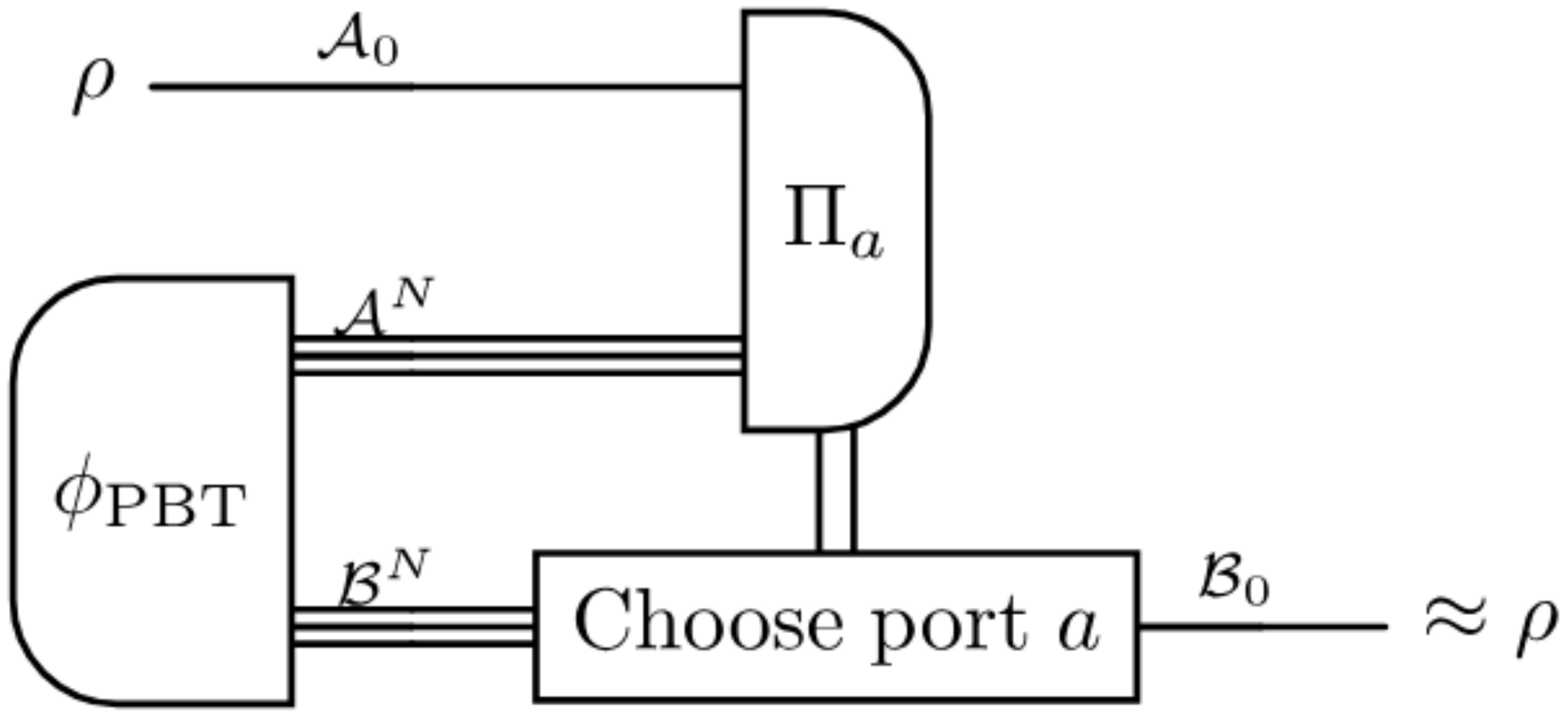
Outline of this talk

- **Definition of the tasks**
- Main result
- Applications
- Proof techniques
- Conclusion

Deterministic port-based teleportation (dPBT)



Quantum circuit



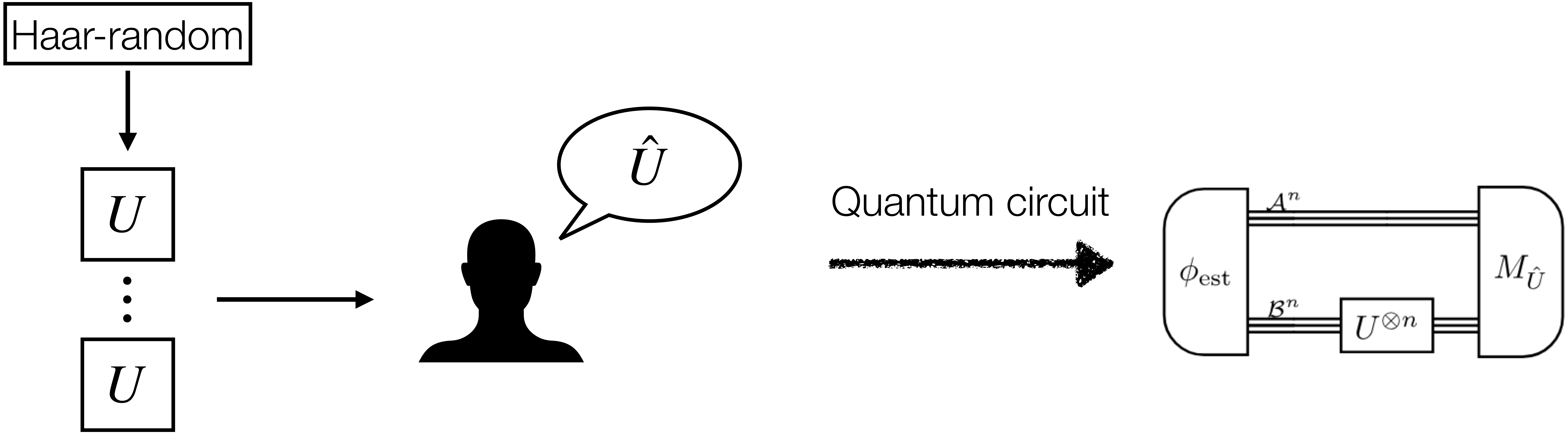
Teleportation channel $\Lambda(\rho) = \sum_{a=1}^{N} \mathrm{Tr}_{\mathcal{A}_0\mathcal{A}^N\overline{\mathcal{B}_a}}[(\Pi_a \otimes I_{\mathcal{B}^N})(\rho \otimes |\phi_{\mathrm{PBT}}\rangle\langle\phi_{\mathrm{PBT}}|)]$

Figure of merit: $F_{\mathrm{PBT}} = f(I_d, \Lambda)$

Channel fidelity

Definition of the tasks

Unitary estimation



Guess probability $p(\hat{U} \,|\, U) = \mathrm{Tr}[M_{\hat{U}} (I_{\mathscr{A}^n} \otimes U_{\mathscr{B}^n}^{\otimes n}) \,|\, \phi_{\mathrm{est}} \rangle \langle \phi_{\mathrm{est}} \,|\, (I_{\mathscr{A}^n} \otimes U_{\mathscr{B}^n}^{\otimes n})^\dagger]$

Figure of merit: $F_{\mathrm{est}} = \int \mathrm{d}U \mathrm{d}\hat{U} p(\hat{U} \,|\, U) \, f(U, \hat{U})$

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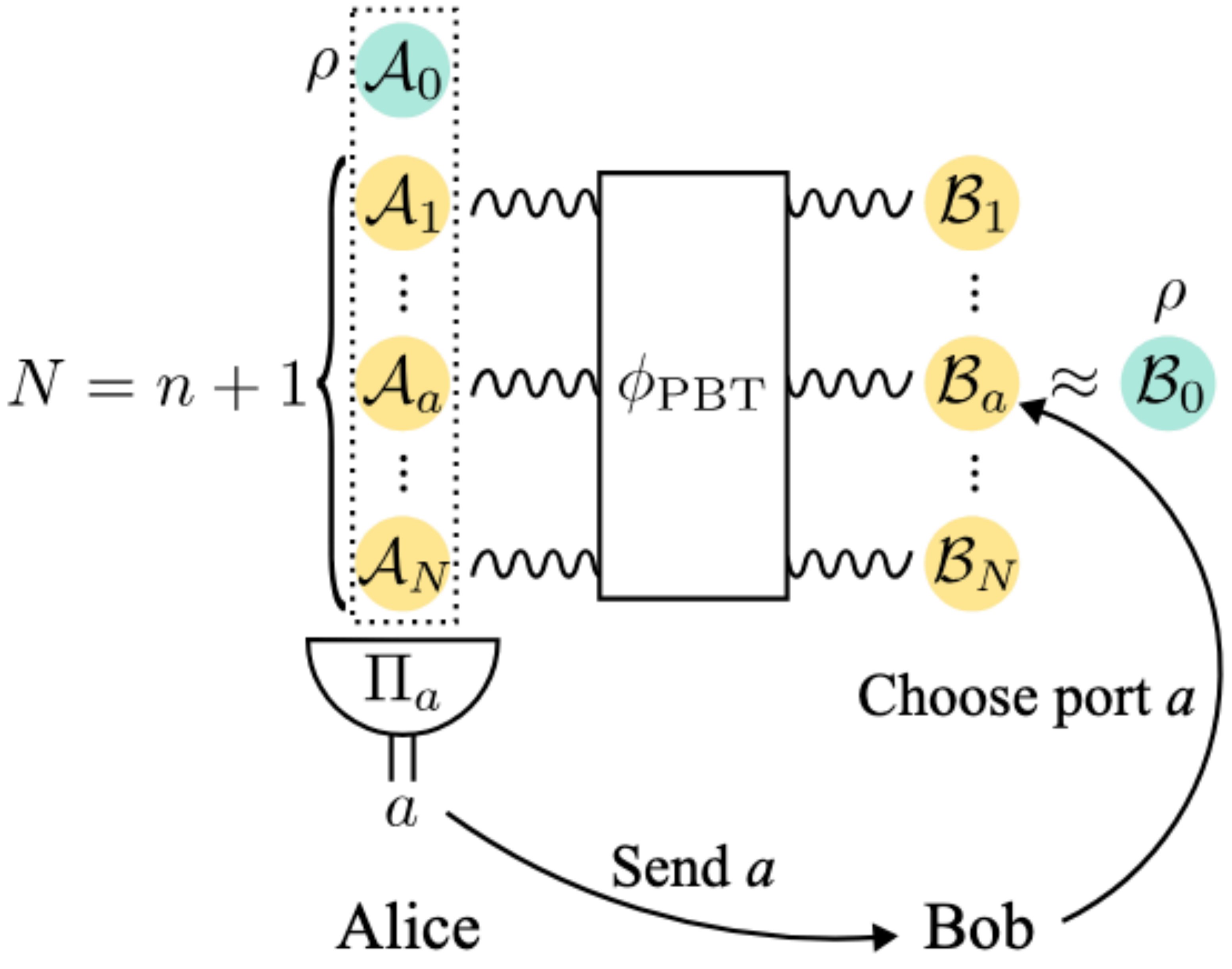
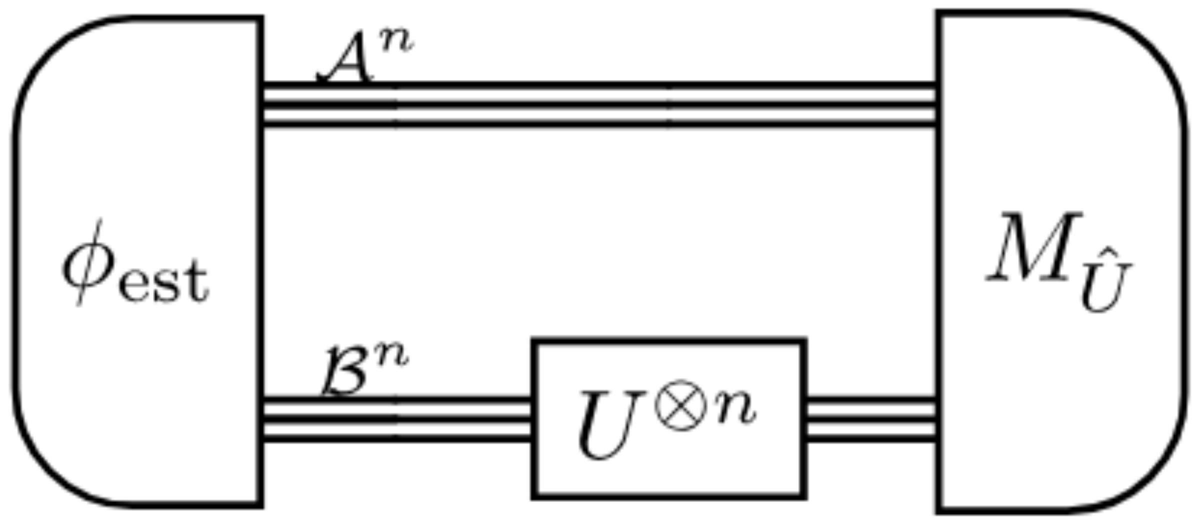
Main result

d-dim. dPBT with $N = n + 1$ ports



d-dim. unitary estimation with n queries



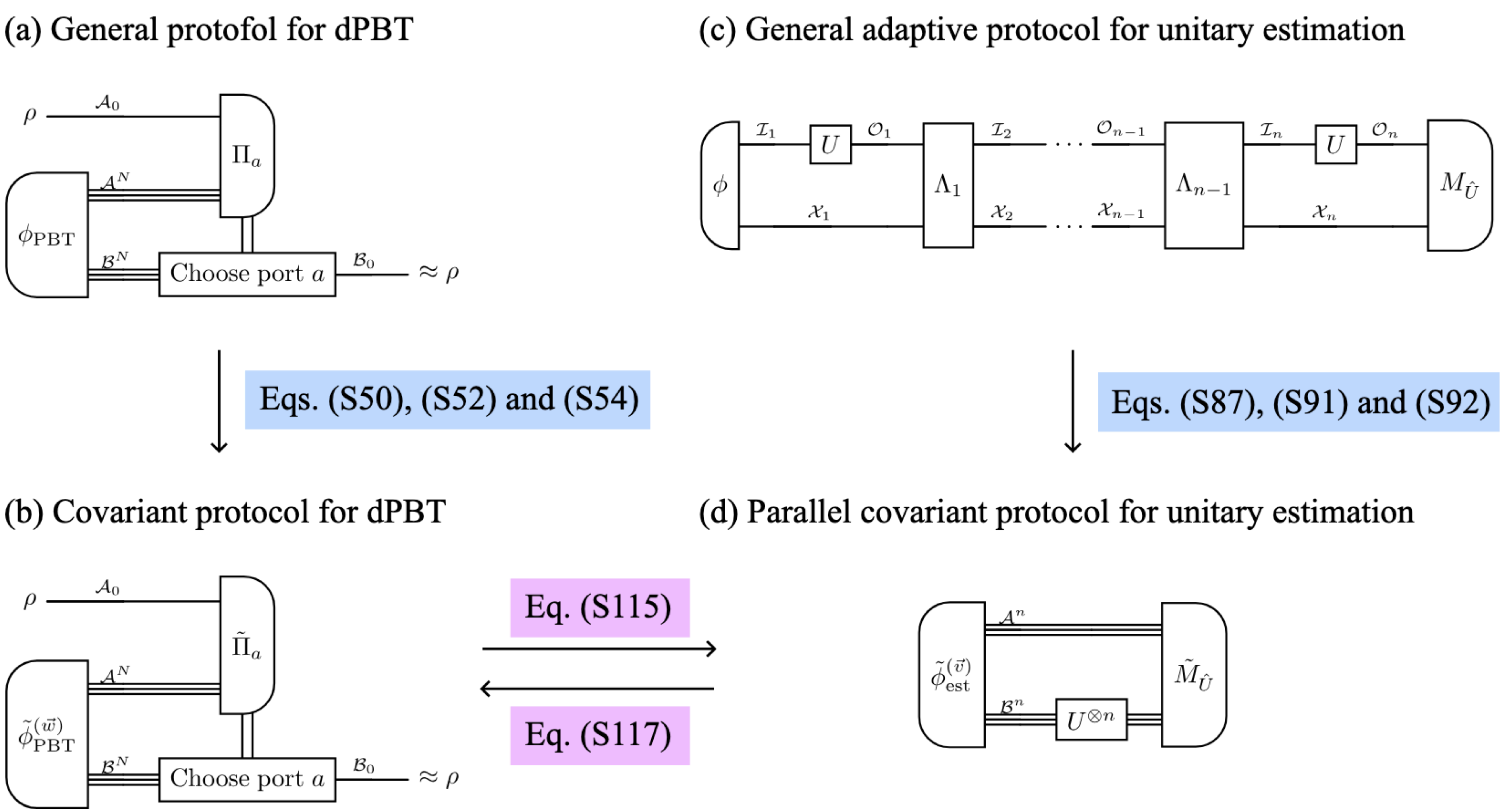
↔

$$F_{\mathrm{PBT}}(n+1, d) = F_{\mathrm{est}}(n, d)$$

✅ Explicit construction from one protocol to the other

One-to-one correspondence between dPBT and unitary estimation

Main result



Conversion of the protocols between dPBT and unitary estimation

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Applications

J. Haah et al. FOCS (2023)
Y. Yang, R. Renner, G. Chiribella, PRL 125, 210501 (2020)
M. Christandl et al. Commun. Math. Phys. 381, 379-451 (2021)

Asymptotically optimal dPBT

Corollary. $F_{\mathrm{PBT}}(N,d) = 1 - \Theta(d^4 N^{-2})$

Optimal unitary estimation obeys Heisenberg limit: $F_{\mathrm{est}}(n,d) = 1 - \Theta(d^4 n^{-2})$

→ Optimal dPBT also satisfies $F_{\mathrm{PBT}}(N,d) = 1 - \Theta(d^4 N^{-2})$

✅ Improved over the previous result $1 - O(d^5 N^{-2}) \leq F_{\mathrm{PBT}}(N,d) \leq 1 - \Omega(d^2 N^{-2})$

✅ Partially solves an open problem to determine $h(d) = \lim_{N\to\infty} [1 - F_{\mathrm{PBT}}(N,d)] N^2 = \Theta(d^4)$

M. Mozrzymas at al. NJP 20, 053006 (2018)

Optimal unitary estimation for $n \leq d - 1$

Corollary. $F_{\text{est}}(n, d) = \dfrac{n+1}{d^2} \quad \text{for } n \leq d - 1$

Optimal PBT for $N \leq d$: $F_{\text{PBT}}(N, d) = \dfrac{N}{d^2}$

→ Optimal unitary estimation for $n \leq d - 1$: $F_{\text{est}}(n, d) = \dfrac{n+1}{d^2}$

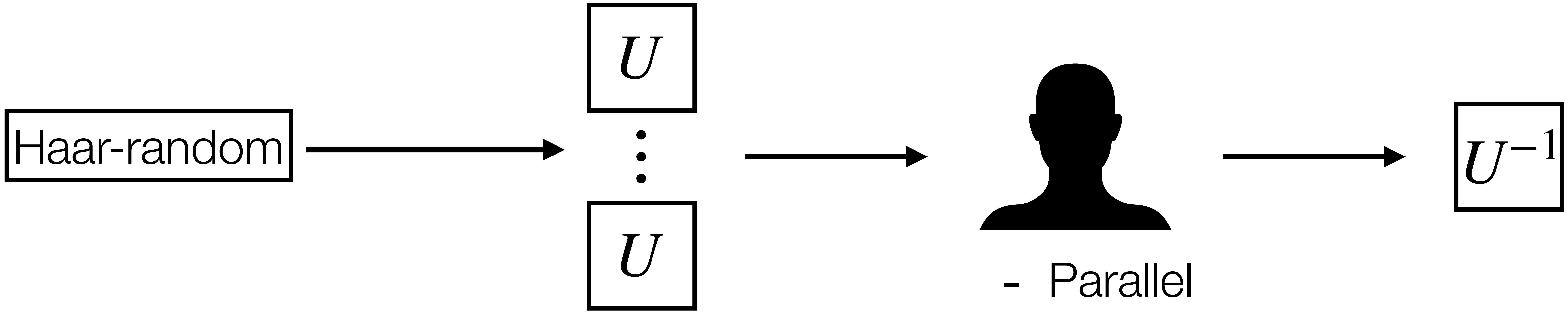
✅ Optimal resource state for PBT = Optimal resource state for unitary estimation

Applications

M. T. Quintino and D. Ebler, Quantum 6, 679 (2022)

Optimal unitary inversion for $n \leq d-1$

Unitary inversion



- Parallel
- Sequential
- Indefinite causal order

Theorem. $F_{\mathrm{inv}}^{(\mathrm{PAR})}(n,d) = F_{\mathrm{inv}}^{(\mathrm{SEQ})}(n,d) = F_{\mathrm{inv}}^{(\mathrm{GEN})}(n,d) = \frac{n+1}{d^2}$ for $n \leq d-1$

✅ Measure-and-prepare is optimal for $n \leq d-1$ even allowing indefinite causal order

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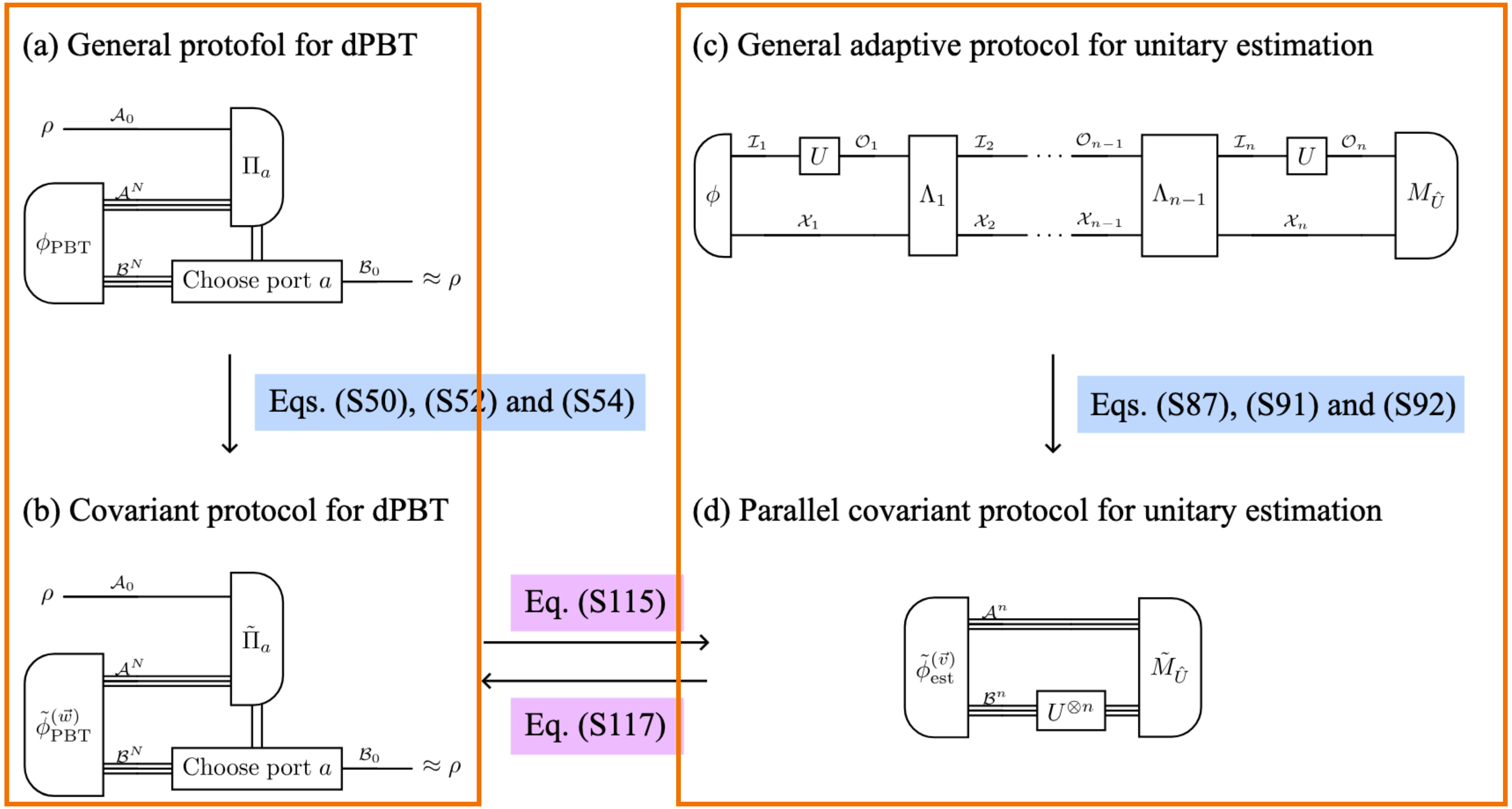
Proof techniques

1. **Symmetry of the problem → Covariant protocol**
2. Conversion between covariant protocols



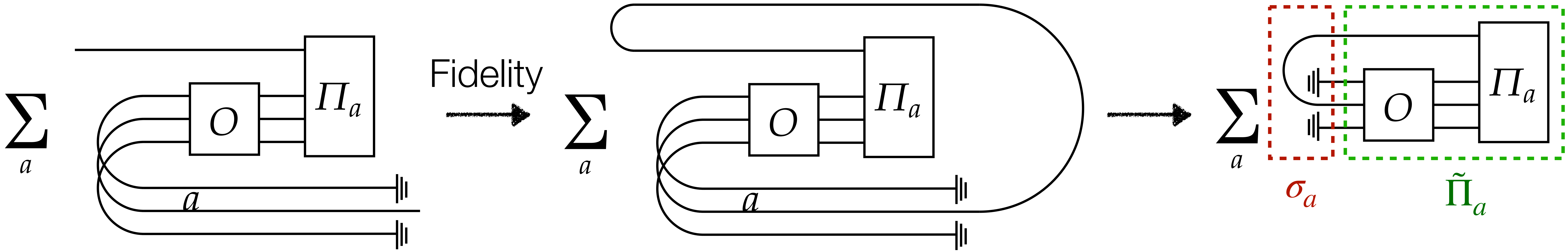
Proof techniques

S. Ishizaka and T. Hiroshima, PRL 101, 240501 (2008)

Optimal fidelity of dPBT given by semidefinite programming (SDP)

W.l.o.g. we can consider $|\phi_{\mathrm{PBT}}\rangle = (O_{\mathscr{A}^N} \otimes I_{\mathscr{B}^N})|\phi^+\rangle_{\mathscr{A}^N\mathscr{B}^N}$,

where $\mathrm{Tr}[O^\dagger O] = d^N$ and $|\phi^+\rangle$ is the maximally entangled state



$$F_{\mathrm{PBT}} = \sum_a \mathrm{Tr}[\tilde{\Pi}_a \sigma_a], \quad \sigma_a := |\phi^+\rangle_{\mathscr{A}_0\mathscr{A}_a} \otimes I_{\overline{\mathscr{A}}_a}/d^{N-2}$$

$$\tilde{\Pi}_a \geq 0, \quad \sum_a \tilde{\Pi}_a = O^\dagger O \otimes I =: X \otimes I$$

→

$$\max \sum_a \mathrm{Tr}[\tilde{\Pi}_a \sigma_a]$$

$$\mathrm{s.t.}\ \tilde{\Pi}_a \geq 0, \ \sum_a \tilde{\Pi}_a = X \otimes I, \ \mathrm{Tr}[X] = d^N$$

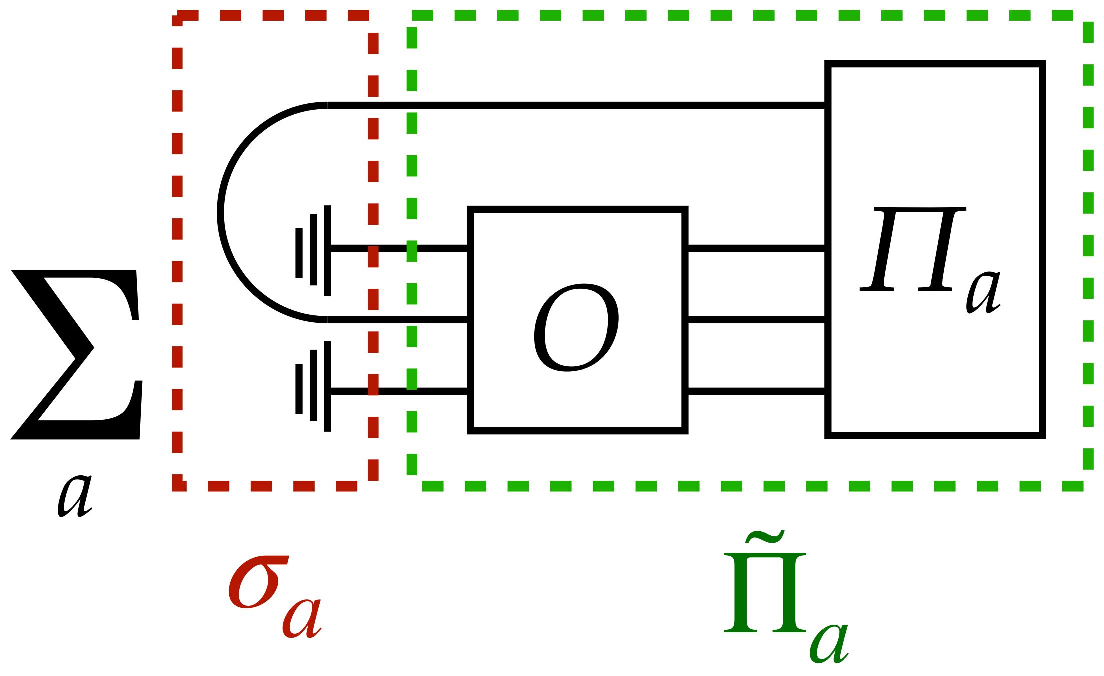
Proof techniques

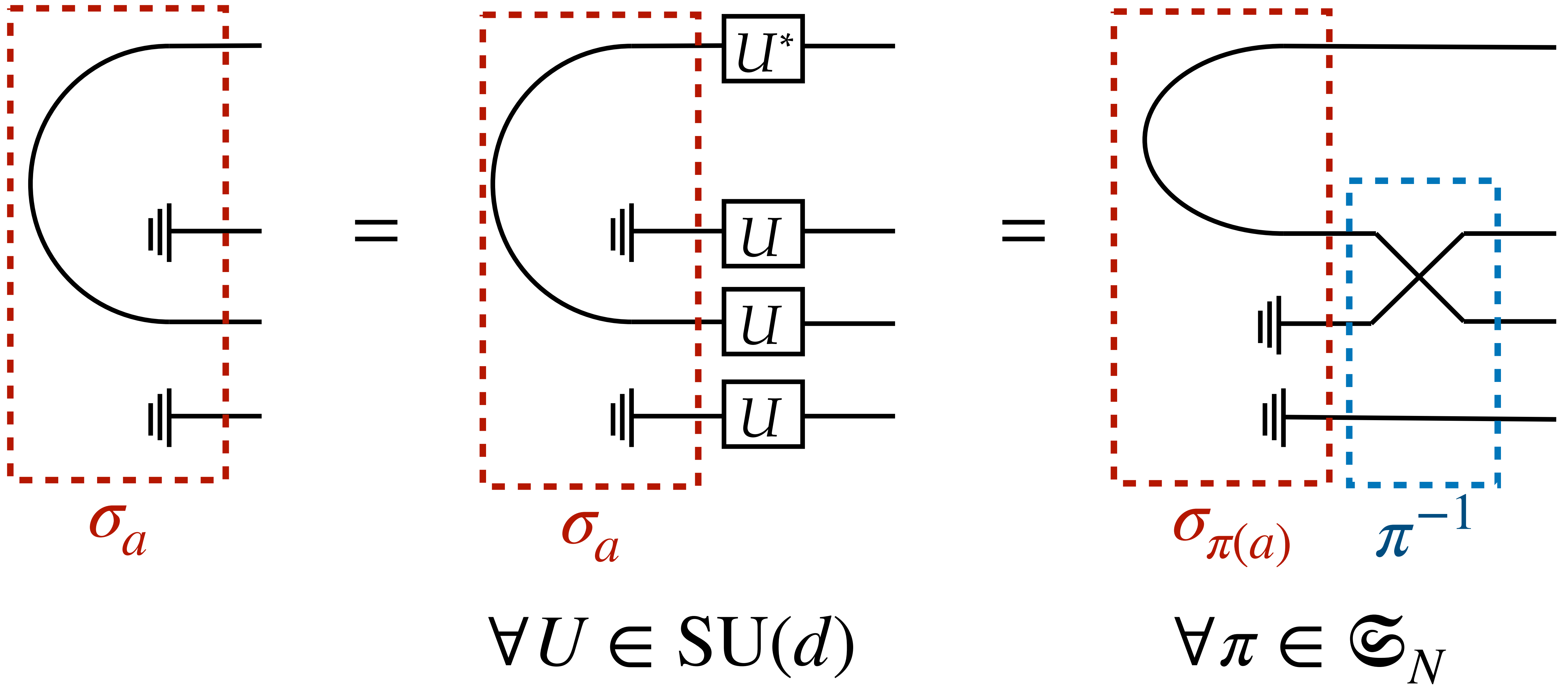
S. Ishizaka and T. Hiroshima, PRL 101, 240501 (2008)
M. Mozrzymas at al. NJP 20, 053006 (2018)
F. Leditzky, Lett. Math. Phys. 112, 98 (2022)

<u>Unitary and symmetric group symmetry of the SDP for dPBT</u>

$$\max \sum_a \mathrm{Tr}[\tilde{\Pi}_a \sigma_a]$$

$$\mathrm{s.t.}\ \tilde{\Pi}_a \geq 0,\ \sum_a \tilde{\Pi}_a = X \otimes I,\ \mathrm{Tr}[X] = d^N$$





$$\longrightarrow \begin{array}{l} [X, U^{\otimes N}] = 0 \quad \forall U \in \mathrm{SU}(d) \\ [X, \pi] = 0 \quad \forall \pi \in \mathfrak{S}_N \end{array}$$

$$\Longrightarrow X = \sum_\mu w_\mu P_\mu$$

P_μ: Young projector

$$(\mathbb{C}^d)^{\otimes N} = \bigoplus_\mu \mathscr{H}_\mu \otimes \mathbb{C}^{m_\mu}$$

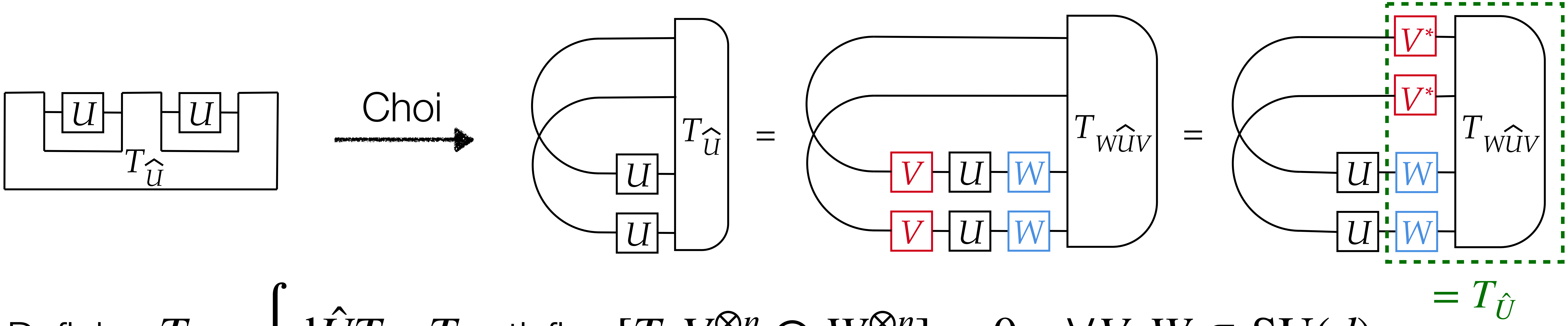
μ: Young diagram

$$F_{\mathrm{PBT}} = \overrightarrow{w}^{\mathsf{T}} M_{\mathrm{PBT}} \overrightarrow{w}$$

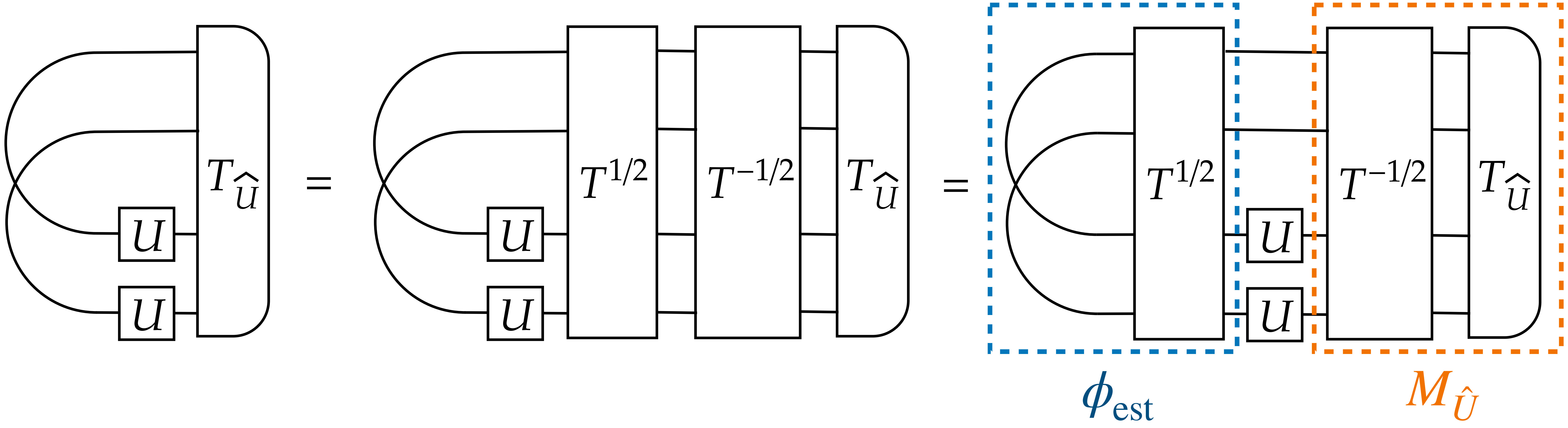
Proof techniques

J. Bavaresco, M. Murao, and M. T. Quintino J. Math. Phys. 63, 042203 (2022)
G. Chiribella, G. M. D'Ariano, and M. F. Sacchi, PRA 72, 042338 (2005)
Y. Yang, R. Renner, G. Chiribella, PRL 125, 210501 (2020)

Unitary group symmetry for unitary estimation



Defining $T := \int \mathrm{d}\hat{U} T_{\hat{U}}$, T satisfies $[T, V^{\otimes n} \otimes W^{\otimes n}] = 0 \quad \forall V, W \in \mathrm{SU}(d)$



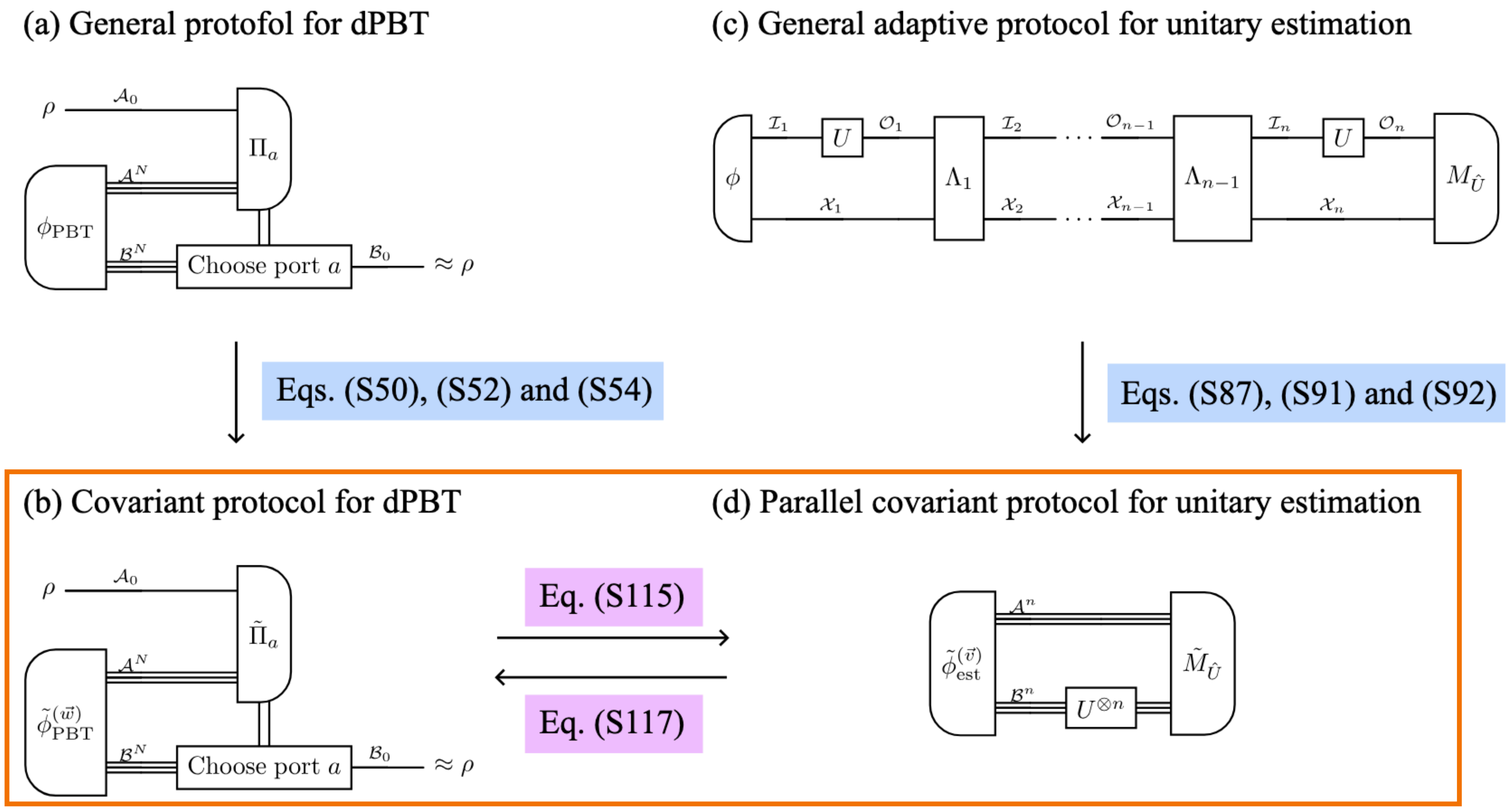
$$\sqrt{T} = \sum_{\alpha} v_\alpha \Pi_\alpha \otimes \Pi_\alpha$$

α: Young diagram

$$F_{\mathrm{est}} = \vec{v}^{\mathsf{T}} M_{\mathrm{est}} \vec{v}$$

Proof techniques

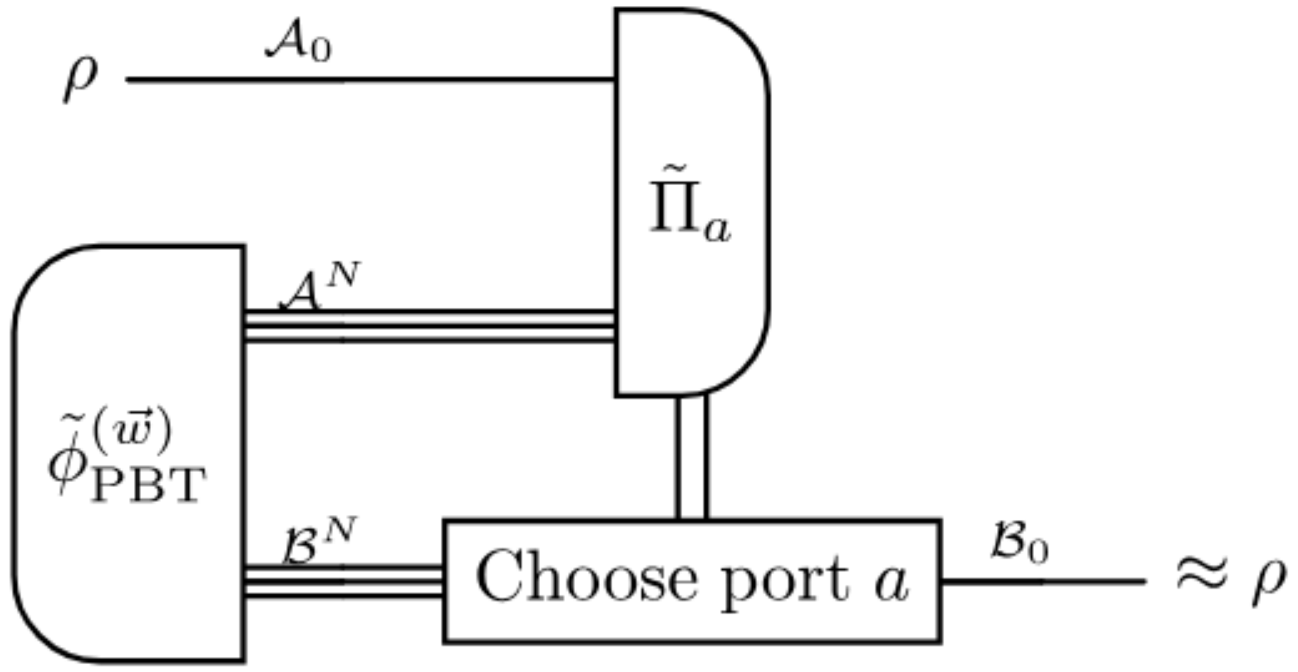
1. Symmetry of the problem → Covariant protocol
2. **Conversion between covariant protocols**



M. Mozrzymas at al. NJP 20, 053006 (2018)
Y. Yang, R. Renner, G. Chiribella, PRL 125, 210501 (2020)

Conversion between the covariant protocols

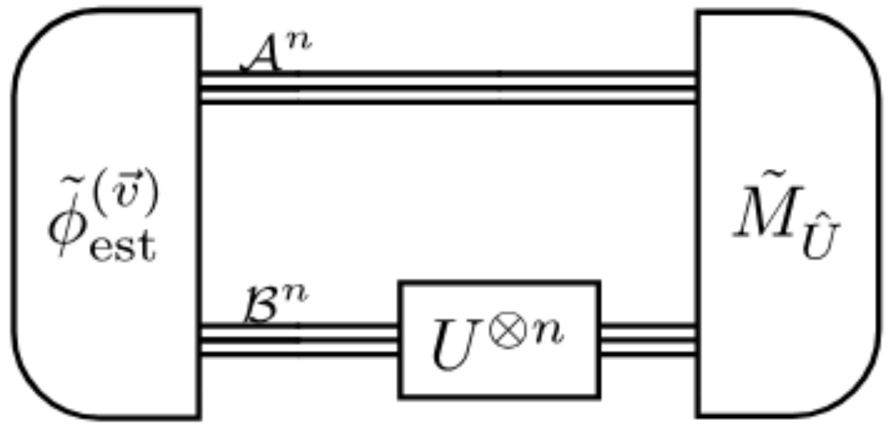
Covariant dPBT protocol



$\vec{v} = R\vec{w}$

$\vec{w} = R^{\mathsf{T}}\vec{v}$

Covariant unitary estimation protocol



$$F_{\mathrm{PBT}} = \vec{w}^{\mathsf{T}} M_{\mathrm{PBT}} \vec{w}$$

$$F_{\mathrm{est}} = \vec{v}^{\mathsf{T}} M_{\mathrm{est}} \vec{v}$$

$$(M_{\mathrm{PBT}})_{\mu\nu} = \frac{R(N-1,d)^{\mathsf{T}} R(N-1,d)}{d^2}$$

$$(M_{\mathrm{est}})_{\alpha\beta} = \frac{R(n,d)R(n,d)^{\mathsf{T}}}{d^2} \qquad [R(n,d)]_{\alpha\mu} = \delta_{\mu\in\alpha+\square}$$

$$\max F_{\mathrm{PBT}} = \max \mathrm{eig}(M_{\mathrm{PBT}}) = \max \mathrm{sing}(R)^2/d^2$$

$$\max F_{\mathrm{est}} = \max \mathrm{eig}(M_{\mathrm{est}}) = \max \mathrm{sing}(R)^2/d^2$$

$$\Longrightarrow \boxed{F_{\mathrm{PBT}}(N,d) = F_{\mathrm{est}}(n,d) \quad \text{for } N = n+1}$$

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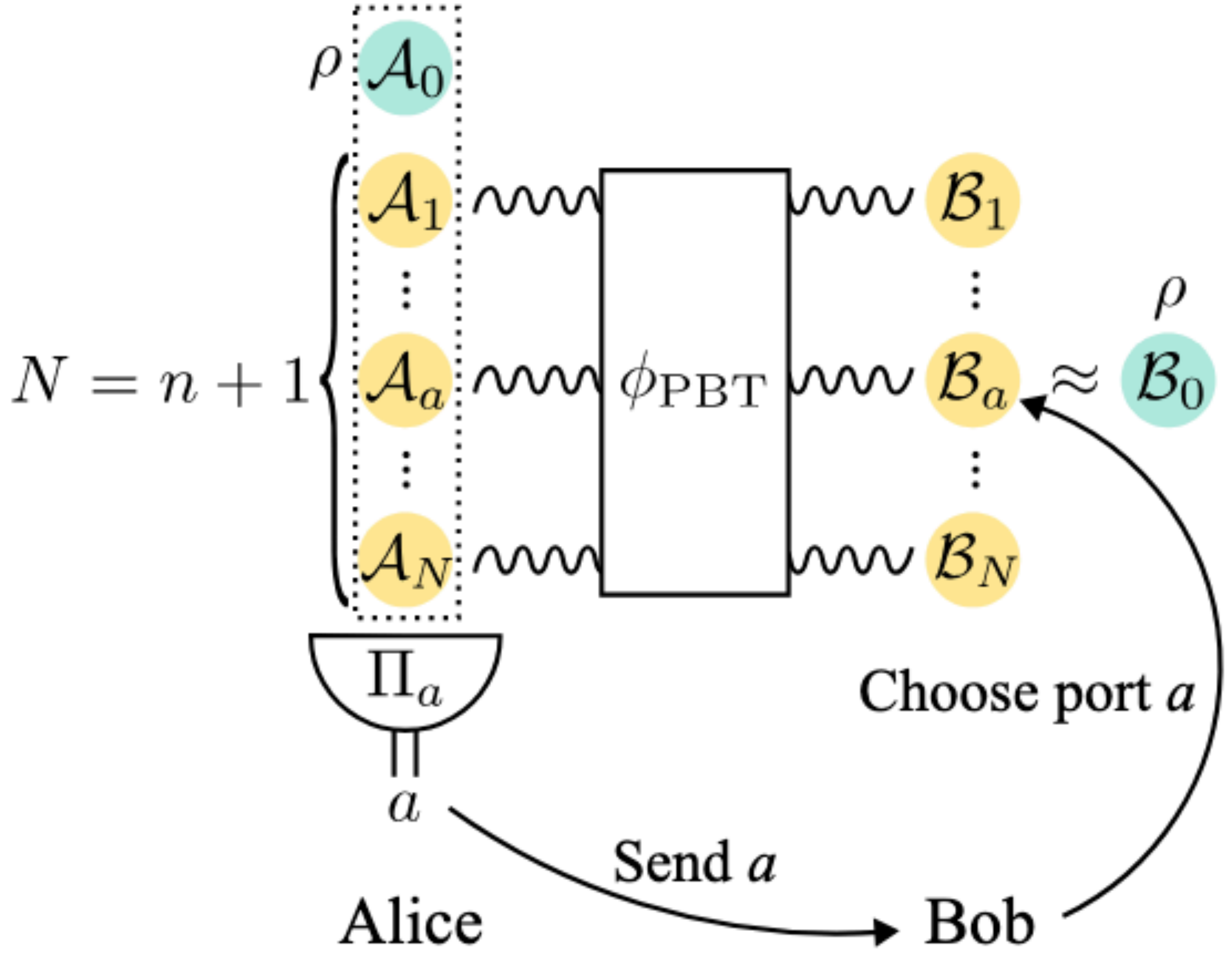
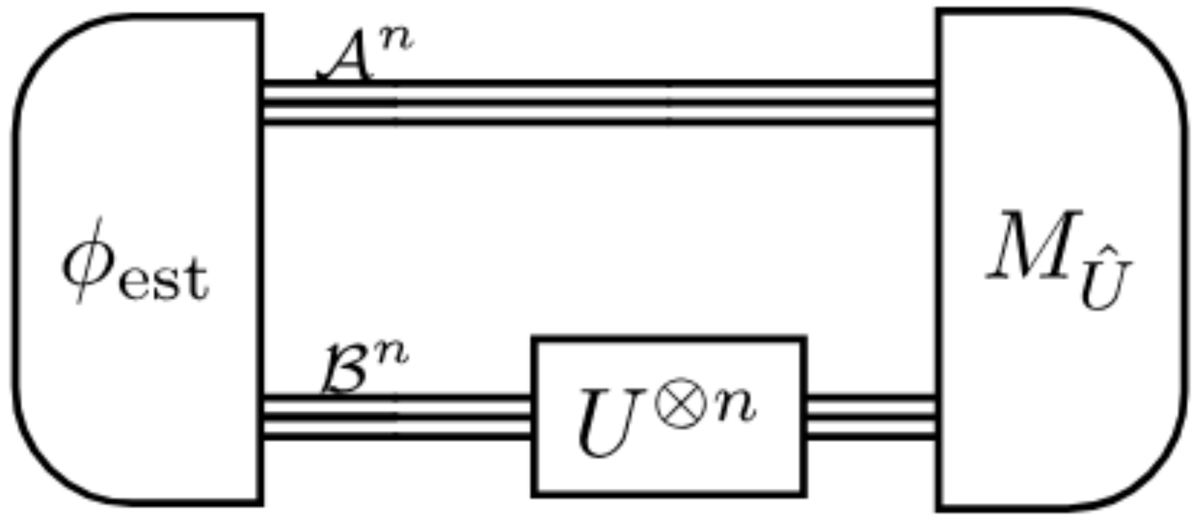
Conclusion

d-dim. dPBT with $N = n + 1$ ports



d-dim. unitary estimation with n queries



$$F_{\mathrm{PBT}}(n+1, d) = F_{\mathrm{est}}(n, d)$$

Take-home: dPBT = unitary estimation!

✅ Explicit construction from one protocol to the other protocol via covariant protocol

❌ Covariant protocol may require more resources than non-covariant one (e.g. #qubits)

E.g. Adaptive unitary estimation using no auxiliary qubits in [J. Haah et al. FOCS (2023)]

Future work: Efficient conversion between unitary estimation and dPBT?

Thank you!