Simulating the quantum switch using causally ordered circuits requires at least an exponential overhead in query complexity

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(*These authors contributed equally)

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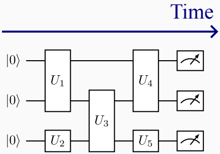
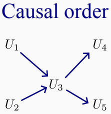
Backgrounds

Causal order in quantum information processing

Causal order in quantum information processing

- Quantum computation





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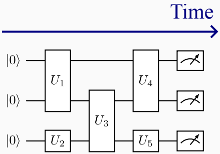
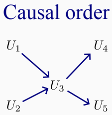
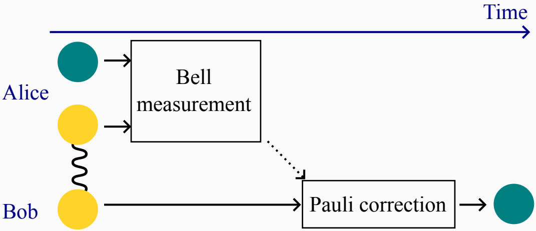


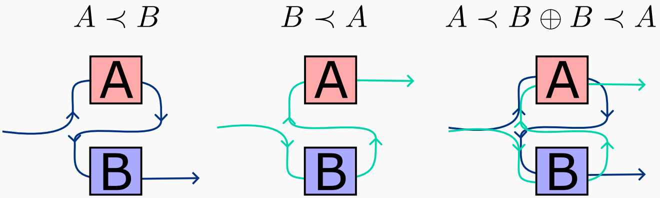


- Quantum communication



Backgrounds

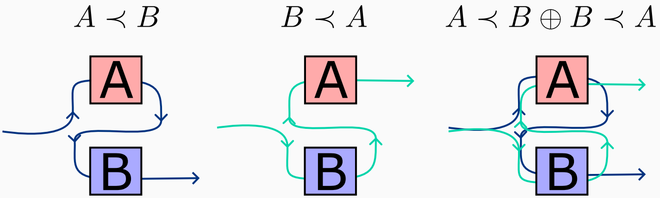
Quantum switch[1]: coherent superposition of causal orders



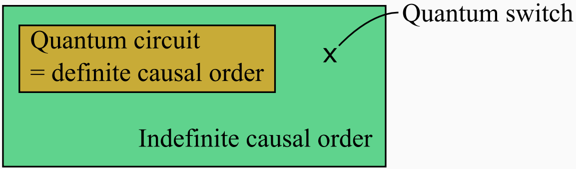
[1] Chiribella et al. 2009; Chiribella et al. 2013.

Backgrounds

Quantum switch[1]: coherent superposition of causal orders



Quantum switch is an example of indefinite causal order



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Backgrounds

Quantum switch: coherent superposition of causal orders

Question

What is a power of the quantum switch in quantum information processing?

Backgrounds

Quantum switch: coherent superposition of causal orders

Question

What is a power of the quantum switch in quantum information processing?

Advantage of the quantum switch on...

- Quantum query complexity
- Quantum communication complexity
- Multipartite games
- Quantum Shannon theory
- Quantum metrology
- Quantum thermodynamics

Advantage of the quantum switch in quantum query complexity

[2] Araújo, Costa, and Brukner 2014.

Advantage of the quantum switch in quantum query complexity

Fourier promise problem[2]

Given a set of $n!$-dimensional unitary gates $\{U_i\}_{i=0}^{n-1}$. Define Π_x for a permutation σ_x of n unitaries by $\Pi_x = U_{\sigma_x(n-1)} \cdots U_{\sigma_x(0)}$.
Promise: $\exists y$ s.t. $\Pi_x = \omega^{xy}\Pi_0 \; \forall x$, where $\omega := e^{2\pi i/n!}$
Task: Decide $y \in \{0, \cdots, n! - 1\}$

[2] Araújo, Costa, and Brukner 2014.

Advantage of the quantum switch in quantum query complexity

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Task: Decide $y \in \{0, \cdots, n!-1\}$

- Quantum n-switch: $O(n)$ calls of unitaries U_i
- Fixed causal order: $\Omega(n^2)$

$\rightarrow$ **Quadratic advantage!**

[2] Araújo, Costa, and Brukner 2014.

Backgrounds

Quantum n-switch

$$\underbrace{A_1 \prec A_2 \prec \cdots \prec A_n \oplus A_2 \prec A_1 \prec \cdots \prec A_n \oplus \cdots}_{n! \text{ combinations}}$$

Quantum switch = Quantum 2-switch

Backgrounds

Exponential separation?

[3] Araújo, Costa, and Brukner 2014.

[4] Chiribella et al. 2009; Chiribella et al. 2013.

Backgrounds

Exponential separation?

For the Fourier promise problem, quadratic separation is optimal.

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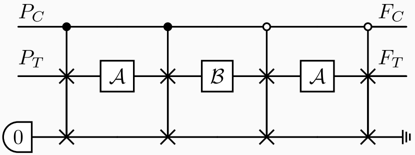
Exponential separation?

For the Fourier promise problem, quadratic separation is optimal.

More generally, the quantum n-switch of unitary channels can be simulated by $O(n^2)$ calls of input channels[3].

$n = 2$ case[4]:



simulates the quantum switch if $\mathcal{A}$ and $\mathcal{B}$ are unitary channels.

[3] Araújo, Costa, and Brukner 2014.

[4] Chiribella et al. 2009; Chiribella et al. 2013.

This talk

Research question

Is there an exponential separation between the quantum switch and a fixed causal order?

[5] Guérin et al. 2016.

This talk

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$\rightarrow$ **Yes!**

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$\rightarrow$ **Yes!**

Remark

- Exponential separation is only known in communication settings[5]

[5] Guérin et al. 2016.

This talk

Research question

Is there an exponential separation between the quantum switch and a fixed causal order?

$\rightarrow$ **Yes!**

Remark

- Exponential separation is only known in communication settings[5]
- We need to extend the input channels to non-unitary channels

[5] Guérin et al. 2016.

Outline of this talk

- Framework: Definition of quantum switch and causal orders
- Problem setting
- Main result: Exponential separation between quantum switch and causally ordered circuit
- Future works

Definition (quantum supermap)

A quantum supermap is a (multi-)linear map of quantum channels.

Definition (quantum switch)

Quantum switch is a 2-slot quantum supermap such that

$$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{U}, \mathcal{V})(\cdot) = S \cdot S^\dagger, \tag{1}$$

$$S = VU \otimes |0\rangle\langle 0| + UV \otimes |1\rangle\langle 1|, \tag{2}$$

for unitary channels $\mathcal{U}$ and $\mathcal{V}$.

Theorem[6]

The above definition uniquely defines the quantum switch.

Framework

Theorem[6]

The above definition uniquely defines the quantum switch.

$$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{A}, \mathcal{B})(\cdot) = \sum_{ij} S_{ij} \cdot S_{ij}^{\dagger}, \tag{1}$$

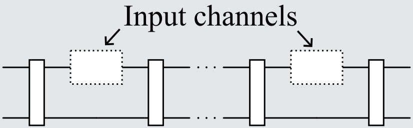
$$S_{ij} = B_j A_i \otimes |0\rangle\langle 0| + A_i B_j \otimes |1\rangle\langle 1| \tag{2}$$

for $\mathcal{A}(\cdot) = \sum_i A_i \cdot A_i^{\dagger}$, $\mathcal{B}(\cdot) = \sum_j B_j \cdot B_j^{\dagger}$

[6] Dong et al. 2023.

Definition (quantum circuit with fixed causal order[7,8])

A quantum circuit with fixed causal order (QC-FO) is a quantum supermap implemented by

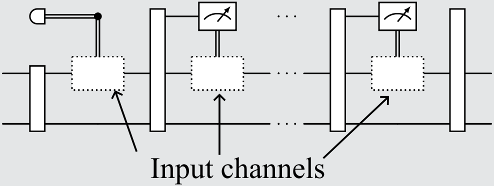


[7] also called quantum comb

[8] Chiribella, D'Ariano, and Perinotti 2008.

Definition (quantum circuit with classical control of the causal order[9])

A quantum circuit with classical control of the causal order (QC-CC) is a quantum supermap implemented by



Remark

QC-CC is believed to be the most general quantum supermap achievable by standard quantum circuits.

[9] Wechs et al. 2021.

Framework

Proposition

quantum switch $\notin$ QC-FO

Framework

Proposition

quantum switch $\notin$ QC-FO

In other words,

Proposition

$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{A}, \mathcal{B})$ cannot be implemented by using a single call of each $\mathcal{A}$ and $\mathcal{B}$ with a fixed causal order

Framework

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quantum switch $\notin$ QC-CC

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$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{A}, \mathcal{B})$ cannot be implemented by using a single call of each $\mathcal{A}$ and $\mathcal{B}$ with a classical control of the causal order

Framework

Proposition

quantum switch $\notin$ QC-CC

In other words,

Proposition

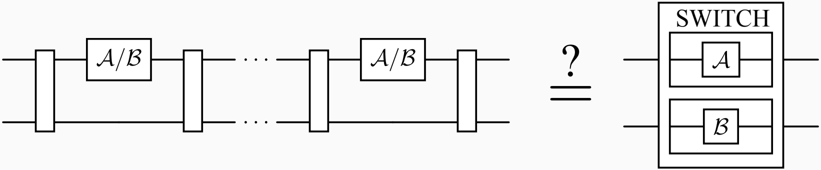
$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{A}, \mathcal{B})$ cannot be implemented by using a single call of each $\mathcal{A}$ and $\mathcal{B}$ with a classical control of the causal order

How about having multiple copies of the input channels?

Problem setting

Question

How many copies of the input quantum channels are needed to simulate the quantum switch using a fixed causal order?



Problem setting

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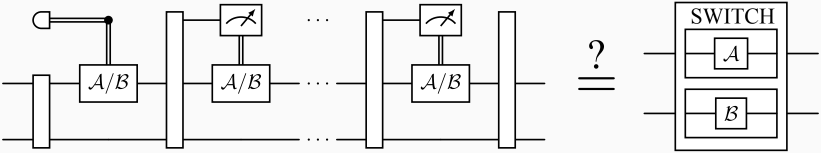
How many copies of the input quantum channels are needed to simulate the quantum switch using a classical control of the causal order?



Main result

Theorem

There is no $(M+1)$-slot supermap with fixed causal order $\mathcal{C}$ satisfying

$$\mathcal{C}(\underbrace{\mathcal{A},\ldots,\mathcal{A}}_{M},\mathcal{B}) = \mathcal{S}_{\texttt{SWITCH}}(\mathcal{A},\mathcal{B}) \tag{3}$$

for all n-qubit channels $\mathcal{A}$ and $\mathcal{B}$, if $M \leq \max(2, 2^n - 1)$.

Main result

Theorem

There is no $(M+1)$-slot supermap with classical control of the causal order $\mathcal{C}$ satisfying

$$\mathcal{C}(\underbrace{\mathcal{A},\ldots,\mathcal{A}}_{M},\mathcal{B}) = \mathcal{S}_{\texttt{SWITCH}}(\mathcal{A},\mathcal{B}) \tag{3}$$

for all mixed unitary n-qubit channels $\mathcal{A}$ and $\mathcal{B}$, if $M \leq \max(2, 2^n - 1)$.

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Remark

- No-go on deterministic and exact simulation

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for all mixed unitary n-qubit channels $\mathcal{A}$ and $\mathcal{B}$, if $M \leq \max(2, 2^n - 1)$.

Remark

- No-go on deterministic and exact simulation
- Multiple copies of only $\mathcal{A}$

Proof sketch

3 steps:

1. Linearity argument
2. Uniqueness
3. Contradiction with QC-FO conditions

Proof sketch

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2. Uniqueness
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First, prepare a nice representation of quantum supermaps

Proof sketch (0. Choi representation)

Definition (Choi representation)

Choi matrix of a linear map $\mathcal{Q} : \mathbb{L}(A) \to \mathbb{L}(B)$:

$$Q := \sum_{ij} |i\rangle\langle j|^A \otimes \mathcal{Q}(|i\rangle\langle j|) \in \mathbb{L}(A \otimes B), \tag{4}$$

where $\{|i\rangle\}$ is the computational basis of $\mathcal{H}^A$ and $\mathbb{L}(A)$ is the set of linear operators on $\mathcal{H}^A$.

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Choi matrix of unitary operation $\mathcal{U}(\cdot) = U \cdot U^\dagger$ is represented as a rank-1 operator

$$|U\rangle\rangle\langle\langle U| \tag{5}$$

where $|U\rangle\rangle$ is a Choi vector defined by $|U\rangle\rangle := \sum_i |i\rangle^A \otimes U\,|i\rangle$.

Proof sketch (0. Choi representation)

Quantum mechanics in the Choi representation

- Composition $\leftrightarrow$ link product
- CPTP map $\mathcal{Q} \leftrightarrow Q \geq 0$ and affine conditions on Q

Link product

Link product of $Q \in \mathbb{L}(A \otimes B)$ and $R \in \mathbb{L}(B \otimes C)$

$$Q * R := \mathrm{Tr}_B[(Q^{AB} \otimes \mathbb{1}^C)^{\mathrm{T}_B}(\mathbb{1}^A \otimes R^{BC})] \tag{6}$$

which satisfies

$$\mathcal{Q}(\rho) = Q * \rho, \tag{7}$$

$$\mathcal{T} = \mathcal{Q} \circ \mathcal{R} \Leftrightarrow T = Q * R. \tag{8}$$

Proof sketch (0. Choi representation)

Quantum supermap in the Choi representation

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Quantum supermap in the Choi representation

QC-FO

$$\mathcal{S}(\mathcal{C}_1, \cdots, \mathcal{C}_n) = \mathcal{V}_M \circ (\mathcal{C}_n \otimes \mathbb{1}) \circ \cdots \circ (\mathcal{C}_1 \otimes \mathbb{1}) \circ \mathcal{V}_0 \tag{9}$$

$$S * (C_1 \otimes \cdots \otimes C_n) = V_M * C_n * \cdots * C_1 * V_0 \tag{10}$$

for $S = V_M * \cdots * V_0$

Proof sketch (0. Choi representation)

Quantum supermap in the Choi representation

QC-FO

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In general, Choi matrix of the output channel $\mathcal{S}(\mathcal{C}_1, \cdots, \mathcal{C}_n)$ is given by $S * (C_1 \otimes \cdots \otimes C_n)$

Proof sketch (0. Choi representation)

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In general, Choi matrix of the output channel $\mathcal{S}(\mathcal{C}_1, \cdots, \mathcal{C}_n)$ is given by $S * (C_1 \otimes \cdots \otimes C_n)$

S is called the Choi matrix of the supermap $\mathcal{S}$

Proof sketch (0. Choi representation)

Characterization of the Choi matrix S of the supermap $\mathcal{S}$

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$\mathcal{S} \in$ QC-FO $\Leftrightarrow S \geq 0 +$ (more strict) affine conditions on S

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$\mathcal{S} \in$ QC-CC $\Leftrightarrow S = \sum_i S_i$, $S_i \geq 0$ + affine conditions on $\{S_i\}_i$

Quantum switch

The Choi matrix $S_{\texttt{SWITCH}}$ of the quantum switch is given by a rank-1 operator

$$S_{\texttt{SWITCH}} = |S_{\texttt{SWITCH}}\rangle\rangle\langle\langle S_{\texttt{SWITCH}}| \tag{11}$$

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2. Uniqueness
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Logical flow:

- Assume that $\mathcal{C}$ simulates the quantum switch
  $\Rightarrow$ Restrict the form of C (steps 1, 2)

Proof sketch

3 steps:

1. Linearity argument
2. Uniqueness
3. Contradiction with QC-FO conditions

Logical flow:

- Assume that $\mathcal{C}$ simulates the quantum switch
  $\Rightarrow$ Restrict the form of C (steps 1, 2)
- The restricted form does not satisfy QC-FO conditions (step 3)

Proof sketch (1. Linearity argument)

For simplicity, we consider $M = 2$ case

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1. Linearity argument

Assume that

$$\mathcal{C}(\mathcal{A}, \mathcal{A}, \mathcal{B}) = \mathcal{S}_{\mathtt{SWITCH}}(\mathcal{A}, \mathcal{B}) \tag{12}$$

for $\mathcal{A} = \mathcal{U}_1, \mathcal{U}_2, \frac{\mathcal{U}_1 + \mathcal{U}_2}{2}$ and $\mathcal{B} = \mathcal{V}$ for unitary operations $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}$,

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$$\mathcal{C}(\mathcal{U}_1, \mathcal{U}_1, \mathcal{V}) = \mathcal{S}_{\texttt{SWITCH}}(\mathcal{U}_1, \mathcal{V}), \tag{13}$$

$$\mathcal{C}(\mathcal{U}_2, \mathcal{U}_2, \mathcal{V}) = \mathcal{S}_{\texttt{SWITCH}}(\mathcal{U}_2, \mathcal{V}), \tag{14}$$

$$\mathcal{C}(\frac{\mathcal{U}_1 + \mathcal{U}_2}{2}, \frac{\mathcal{U}_1 + \mathcal{U}_2}{2}, \mathcal{V}) = \mathcal{S}_{\texttt{SWITCH}}(\frac{\mathcal{U}_1 + \mathcal{U}_2}{2}, \mathcal{V}), \tag{15}$$

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For simplicity, we consider $M = 2$ case

1. Linearity argument

Assume that

$$\mathcal{C}(\mathcal{A},\mathcal{A},\mathcal{B}) = \mathcal{S}_{\mathtt{SWITCH}}(\mathcal{A},\mathcal{B}) \tag{12}$$

for $\mathcal{A} = \mathcal{U}_1, \mathcal{U}_2, \frac{\mathcal{U}_1+\mathcal{U}_2}{2}$ and $\mathcal{B} = \mathcal{V}$ for unitary operations $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}$,

$$\mathcal{C}(\mathcal{U}_1,\mathcal{U}_1,\mathcal{V}) = \mathcal{S}_{\mathtt{SWITCH}}(\mathcal{U}_1,\mathcal{V}), \tag{13}$$

$$\mathcal{C}(\mathcal{U}_2,\mathcal{U}_2,\mathcal{V}) = \mathcal{S}_{\mathtt{SWITCH}}(\mathcal{U}_2,\mathcal{V}), \tag{14}$$

$$\mathcal{C}(\frac{\mathcal{U}_1+\mathcal{U}_2}{2},\frac{\mathcal{U}_1+\mathcal{U}_2}{2},\mathcal{V}) = \mathcal{S}_{\mathtt{SWITCH}}(\frac{\mathcal{U}_1+\mathcal{U}_2}{2},\mathcal{V}), \tag{15}$$

thus

$$\mathcal{C}(\mathcal{U}_1,\mathcal{U}_2,\mathcal{V}) + \mathcal{C}(\mathcal{U}_2,\mathcal{U}_1,\mathcal{V}) = \mathcal{S}_{\mathtt{SWITCH}}(\mathcal{U}_1+\mathcal{U}_2,\mathcal{V}). \tag{16}$$

Proof sketch (1. Linearity argument)

In terms of Choi:

$$\begin{aligned} & C * (|U_1\rangle\!\rangle\langle\!\langle U_1| \otimes |U_2\rangle\!\rangle\langle\!\langle U_2| \otimes |V\rangle\!\rangle\langle\!\langle V|) \\ & + C * (|U_2\rangle\!\rangle\langle\!\langle U_2| \otimes |U_1\rangle\!\rangle\langle\!\langle U_1| \otimes |V\rangle\!\rangle\langle\!\langle V|) \\ & = |S_{\texttt{SWITCH}}\rangle\!\rangle\langle\!\langle S_{\texttt{SWITCH}}| * [(|U_1\rangle\!\rangle\langle\!\langle U_1| + |U_2\rangle\!\rangle\langle\!\langle U_2|) \otimes |V\rangle\!\rangle\langle\!\langle V|] \end{aligned} \tag{17}$$

Proof sketch (1. Linearity argument)

In terms of Choi:

$$
\begin{aligned}
& C * (|U_1\rangle\!\rangle\langle\!\langle U_1| \otimes |U_2\rangle\!\rangle\langle\!\langle U_2| \otimes |V\rangle\!\rangle\langle\!\langle V|) \\
& + C * (|U_2\rangle\!\rangle\langle\!\langle U_2| \otimes |U_1\rangle\!\rangle\langle\!\langle U_1| \otimes |V\rangle\!\rangle\langle\!\langle V|) \leftarrow \text{positive} \\
& = |S_{\texttt{SWITCH}}\rangle\!\rangle\langle\!\langle S_{\texttt{SWITCH}}| * [(|U_1\rangle\!\rangle\langle\!\langle U_1| + |U_2\rangle\!\rangle\langle\!\langle U_2|) \otimes |V\rangle\!\rangle\langle\!\langle V|] \qquad (17)
\end{aligned}
$$

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Since $C \geq 0$, C is written as $C = \sum_i |C_i\rangle\rangle\langle\langle C_i|$. Then

$$\begin{aligned} &|C_i\rangle\rangle\langle\langle C_i| * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|) \\ &\leq C * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|) \\ &\leq |S_{\mathtt{SWITCH}}\rangle\rangle\langle\langle S_{\mathtt{SWITCH}}| * [(|U_1\rangle\rangle\langle\langle U_1| + |U_2\rangle\rangle\langle\langle U_2|) \otimes |V\rangle\rangle\langle\langle V|]. \end{aligned} \quad (18)$$

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\begin{aligned}
&C * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|) \\
&\leq |S_{\texttt{SWITCH}}\rangle\rangle\langle\langle S_{\texttt{SWITCH}}| * [(|U_1\rangle\rangle\langle\langle U_1| + |U_2\rangle\rangle\langle\langle U_2|) \otimes |V\rangle\rangle\langle\langle V|]
\end{aligned} \tag{17}
$$

Since $C \geq 0$, C is written as $C = \sum_i |C_i\rangle\rangle\langle\langle C_i|$. Then

$$
\begin{aligned}
&|C_i\rangle\rangle\langle\langle C_i| * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|) \\
&\leq C * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|) \\
&\leq |S_{\texttt{SWITCH}}\rangle\rangle\langle\langle S_{\texttt{SWITCH}}| * [(|U_1\rangle\rangle\langle\langle U_1| + |U_2\rangle\rangle\langle\langle U_2|) \otimes |V\rangle\rangle\langle\langle V|].
\end{aligned} \tag{18}
$$

Thus,

$$
\begin{aligned}
&|C_i\rangle\rangle * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) \\
&= \sum_{l=1}^{2} p_i^{(l)}(U_1, U_2, V) |S_{\texttt{SWITCH}}\rangle\rangle * (|U_l\rangle\rangle \otimes |V\rangle\rangle)
\end{aligned} \tag{19}
$$

Proof sketch (2. Uniqueness)

2. Uniqueness

[10] Odake, Yoshida, and Murao 2024 (Poster in Tuesday).

Proof sketch (2. Uniqueness)

2. Uniqueness

$$\begin{aligned}
&|C_i\rangle\rangle * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) \\
&= \sum_{l=1}^{2} p_i^{(l)}(U_1, U_2, V) |S_{\texttt{SWITCH}}\rangle\rangle * (|U_l\rangle\rangle \otimes |V\rangle\rangle) \qquad (20)
\end{aligned}$$

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Proof sketch (2. Uniqueness)

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$$
\begin{aligned}
&|C_i\rangle\!\rangle * (|U_1\rangle\!\rangle \otimes |U_2\rangle\!\rangle \otimes |V\rangle\!\rangle) \\
&= \sum_{l=1}^{2} p_i^{(l)}(U_1, U_2, V) |S_{\texttt{SWITCH}}\rangle\!\rangle * (|U_l\rangle\!\rangle \otimes |V\rangle\!\rangle)
\end{aligned} \tag{20}
$$

holds if

$$
|C_i\rangle\!\rangle = \sum_{l=1}^{2} |S_{\texttt{SWITCH}}\rangle\!\rangle^{l3} \otimes |p_i^{(l)}\rangle\!\rangle^{\bar{l}} \tag{21}
$$

with $p_i^{(l)}(U_1, U_2, V) = |p_i^{(l)}\rangle\!\rangle * |U_{\bar{l}}\rangle\!\rangle$.

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$$
\begin{aligned}
&|C_i\rangle\!\rangle * (|U_1\rangle\!\rangle \otimes |U_2\rangle\!\rangle \otimes |V\rangle\!\rangle) \\
&= \sum_{l=1}^{2} p_i^{(l)}(U_1, U_2, V)|S_{\texttt{SWITCH}}\rangle\!\rangle * (|U_l\rangle\!\rangle \otimes |V\rangle\!\rangle) \qquad (20)
\end{aligned}
$$

holds if

$$
|C_i\rangle\!\rangle = \sum_{l=1}^{2} |S_{\texttt{SWITCH}}\rangle\!\rangle^{l3} \otimes |p_i^{(l)}\rangle\!\rangle^{\bar{l}} \qquad (21)
$$

with $p_i^{(l)}(U_1, U_2, V) = |p_i^{(l)}\rangle\!\rangle * |U_{\bar{l}}\rangle\!\rangle$.

We show the converse using a differentiation technique[10].

[10] Odake, Yoshida, and Murao 2024 (Poster in Tuesday).

Proof sketch (2. Uniqueness)

We show that $p_i^{(I)}(U_1, U_2, V)$ is

1. linear with respect to $U_{\bar{I}}$
2. independent of U_I and V

by differentiating with respect to U_1, U_2, V

Proof sketch (2. Uniqueness)

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$\Rightarrow \exists |p_i^{(I)}\rangle\rangle$ such that $p_i^{(I)}(U_1, U_2, V) = |p_i^{(I)}\rangle\rangle * |U_{\bar{I}}\rangle\rangle$

Proof sketch (2. Uniqueness)

We show that $p_i^{(l)}(U_1, U_2, V)$ is

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$\Rightarrow \exists |p_i^{(l)}\rangle\rangle$ such that $p_i^{(l)}(U_1, U_2, V) = |p_i^{(l)}\rangle\rangle * |U_{\bar{l}}\rangle\rangle$

$$
\begin{aligned}
&|C_i\rangle\rangle * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) \\
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\end{aligned}
\tag{22}
$$

Proof sketch (2. Uniqueness)

We show that $p_i^{(l)}(U_1, U_2, V)$ is

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$\Rightarrow \exists |p_i^{(l)}\rangle\rangle$ such that $p_i^{(l)}(U_1, U_2, V) = |p_i^{(l)}\rangle\rangle * |U_{\bar{l}}\rangle\rangle$

$$\begin{aligned} &|C_i\rangle\rangle * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) \\ &= \sum_{l=1}^{2} (|S_{\texttt{SWITCH}}\rangle\rangle^{l3} \otimes |p_i^{(l)}\rangle\rangle^{\bar{l}}) * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) \end{aligned} \tag{22}$$

$\Rightarrow |C_i\rangle\rangle = \sum_{l=1}^{2} |S_{\texttt{SWITCH}}\rangle\rangle^{l3} \otimes |p_i^{(l)}\rangle\rangle^{\bar{l}}$

Proof sketch (3. Contradiction with QC-FO conditions)

3. Contradiction with QC-FO conditions

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If $\mathcal{C}$ is QC-FO, then C should satisfy affine conditions

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3. Contradiction with QC-FO conditions

If $\mathcal{C}$ is QC-FO, then C should satisfy affine conditions

As shown in steps 1 and 2, if $\mathcal{C}$ simulates the quantum switch, then $C = \sum_i |C_i\rangle\!\rangle\langle\!\langle C_i|$ for $|C_i\rangle\!\rangle = \sum_{l=1}^{2} |S_{\mathtt{SWITCH}}\rangle\!\rangle^{l3} \otimes |p_i^{(l)}\rangle\!\rangle^{\bar{l}}$

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$\rightarrow$ Contradiction!!

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$\rightarrow$ Contradiction!!

Similar argument for QC-CC

Future works

- Approximate or probabilistic settings?

[11] Bavaresco et al., In preparation

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- More relaxed settings (e.g. only simulating reduced quantum switch)?

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- More relaxed settings (e.g. only simulating reduced quantum switch)?
- Multiple copies of both input channels $\mathcal{A}$ and $\mathcal{B}$?

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Future works

- Approximate or probabilistic settings?
- More relaxed settings (e.g. only simulating reduced quantum switch)?
- Multiple copies of both input channels $\mathcal{A}$ and $\mathcal{B}$?
- Is it possible to exactly simulate a quantum switch by using exponentially many copies of input channels?

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Future works

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- Is it possible to exactly simulate a quantum switch by using exponentially many copies of input channels?

$\rightarrow$ We also investigate these questions by numerical simulations in a companion paper[11].

[11] Bavaresco et al., In preparation

Conclusion

Take home

Simulation of the quantum switch is (at least) exponentially hard

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Proof technique

Linear algebra + differentiation technique

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Thank you!